Consistent Specification Test for the Quantile Autoregression With No Omitted Latent Factors*

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Abstract

This paper proposes a test for the joint hypothesis of correct dynamic specification and no omitted latent factors for the Quantile Autoregression. If the composite null is rejected we proceed to disentangle the cause of rejection, i.e., dynamic misspecification or an omitted variable. We establish the asymptotic distribution of the test statistics under fairly weak conditions and show that factor estimation error is negligible. A Monte Carlo study shows that the suggested tests have good finite sample properties. Finally, we undertake an empirical illustration of modelling growth and inflation in the United Kingdom, where we find evidence that factor augmented models are correctly specified in contrast with their non-augmented counterparts when it comes to GDP growth, while also exploring the asymmetric behaviour of the growth and inflation distribution.

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1 Introduction

Quantile regression (QR) enables the analysis of a continuous range of conditional quantile functions, providing a more complete picture of the conditional dependence structure of the variables examined, rather than a single measure of conditional location. As the awareness of the importance of data heterogeneity increases, quantile regression has become even more relevant (Koenker, 2017). At the same time, the recent availability of large datasets has generated interest in models with many possible predictors. Inference methods overcoming the curse of dimensionality have become increasingly popular and factors, in particular, have been proven useful in overcoming the limited information bias. This arises from the fact that the information set of decision makers is sufficiently larger than the information set captured by conventional empirical models.

The intersection of latent factors with quantile regression models is fairly recent. Ando & Tsay (2011) have considered a quantile regression model with factor-augmented predictors, whose effect is allowed to vary across the different quantiles. Their study models the quantile structure in a cross-sectional context. More recently, Ando & Bai (2020) introduced a new procedure for analysing the quantile co-movement of a large number of time series based on a large scale panel data model with factor structures. In their study the latent factors are allowed to vary across the different quantiles of the variables from which they are extracted and, as such, their model is a quantile factor model. Similarly, Chen *et al.* (2019) estimate scale-shifting factors and quantile dependent loadings, thus factors may shift characteristics (moments or quantiles) of the distribution of the set of directly observable measures, other than its mean, and factor loadings are allowed to vary across the distributional characteristics of each variable. We add to this relevant literature by using mean shifting factors, that is, factors that are only allowed to alter the location of the observable measures as a method of dimension reduction, and use these latent factors as additional regressors in quantile autoregressive models.

Meanwhile, in the empirical literature, the majority of the work has examined whether univariate regression models should be augmented with factors in estimates and forecasts of the conditional mean (Stock & Watson, 2002; Bernanke *et al.*, 2005). However, point estimates and forecasts for the conditional mean of macroeconomic variables, like growth and inflation, ignore the risks around this central estimates (Adrian *et al.*, 2019). Furthermore, over the recent years, the tendency of policymakers to focus more on the downside risk of GDP growth demonstrates the need to analyse the conditional quantiles of GDP growth and examine how they should be modelled individually. Similarly, as supported by Levin & Piger (2002) and Angeloni *et al.* (2006), the inflation rate is one of the most important variables, due to its dominant role in many macroeconomic models and, as argued by, inter alia, Henry & Shields (2004), the dynamic behaviour of the inflation rate has a number of economic implications. In order to correct for this omission, we trace the conditional distributions of GDP growth and CPI inflation in the United Kingdom by estimating several conditional quantiles.

In this paper, therefore, our contribution to the existing literature is twofold. Firstly, we allow for the interaction of mean shifting factors (summarising a larger information set) with quantile autoregressive models and, furthermore, as a result, obtain a trace of the conditional distribution of a variable of interest. The latter enables us to assess the upside and downside risks present for the dependent variable. We focus on GDP growth and CPI inflation in the United Kingdom and find that quantile autoregressive models for GDP growth should include latent factors, as a way to summarise multiple macroeconomic variables. Such factors have non-uniform effects on different quantiles of the GDP growth distribution. However, these latent factors do not carry relevant information for modelling the CPI inflation rate distribution. We also find evidence of a decrease in the asymmetry of UK GDP growth following the recent financial crisis and significant time variation in downside risk. Meanwhile, in the case of inflation upside risk has varied significantly over the years despite a rather stable central tendency.

Nevertheless, for any post estimation inference to be valid, the correct specification of the empirical model used needs to be assessed. Therefore, we also propose a test for the joint hypothesis of correct dynamic specification and no omitted latent factor for the quantile autoregression (QAR), as is outlined in Koenker & Xiao (2006). The test we suggest is related to the conditional moment tests of Bierens (1982, 1990), as well as the specification test of parametric quantile models of Escanciano & Velasco (2010).[1] In practice, we propose two tests: the first can be used to determine if the quantile autoregression is correctly specified, conditional on all available past information of the dependent variable and latent factors. In cases where the null hypothesis fails to reject, we have no evidence that the QAR is not correctly specified. If, however, the null hypothesis is rejected, we perform a second test in order to determine whether the misspecification arises because latent factors are an omitted variable or because the model is dynamically misspecified. The second test therefore involves testing the null hypothesis of the Factor-Augmented QAR (FA-QAR) being correctly specified. Valid asymptotic critical values are obtained via a bootstrap procedure based on resampling functions of the entire history of available information as in Corradi *et al.* (2009).

The remainder of the paper is organised as follows. In Section 2, we outline the framework and describe the testing procedure in order to examine whether a Quantile Autoregression should be augmented with latent factors. We define the test statistics and study the asymptotic properties of the suggested statistics. Section 3 demonstrates some finite sample performance results based on a limited Monte Carlo simulation. In Section 4, we examine the distributions of UK GDP growth

[1] Though similar, the work by Escanciano & Velasco (2010) cannot account for the factor estimation error present in our context due to the use of unobservable factors that need to be estimated a priori.

and CPI inflation based on the use of a large-scale macroeconomic dataset and discuss the empirical findings. Concluding remarks are given in Section 5 and information regarding proofs is referred to an appendix.

2 The Framework

2.1 Test Statistics

We begin by outlining the factor model used in the sequel. Let

$$X_t = \Lambda_t F_t + e_t \tag{1}$$

where X_t is an $N \times 1$ vector of observable variables characterising the economy, Λ_t is an $N \times k$ matrix of factor loadings, F_t is a $k \times 1$ vector of the k latent common factors and e_t is an $N \times 1$ vector of idiosyncratic disturbances. The errors are allowed to be both serially and (weakly) cross sectionally correlated.[2]

As in Stock & Watson (2002), factors are extracted via the principle components approach and the estimated factors and estimated factor loadings are defined as:

$$(\hat{F}, \hat{\Lambda}) = \arg\min_{F,\Lambda} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (X_{it} - \Lambda_i F_t)^2.$$

The resulting principal components estimator of F is then $\hat{F} = \frac{X'\hat{\Lambda}}{N}$, where $\hat{\Lambda}$ is set equal to the eigenvectors of $X'X$ corresponding to its k largest eigenvalues. In the remainder of this paper the number of factors k would remain fixed and can be estimated using the information criteria outlined in Bai & Ng (2002) who take into account the sample size both in the cross-section and time-series dimensions.

Suppose we observe a real-valued dependent variable y_t and a high-dimensional information vector X_t. Due to the curse of dimensionality we wish to reduce the dimension of X_t with the use of factors, as a way to summarise all the available information. The desirable information vector is therefore $I_{t-1} = (Y'_{t-1}, F'_{t-1}) \in \Re^d$, $d = (p+1) + k$, where $F_{t-1} = (F_{1,t-1}, ..., F_{k,t-1}) \in \Re^k, k \in \aleph$, is the vector of latent factors and $Y_{t-1} = (1, y_{t-1}, ..., y_{t-p}) \in \Re^{p+1}$, where A' denotes the transpose of A. In practice, the vector F_{t-1} is unobservable, therefore, we replace the infeasible information

[2] Weak correlations are allowed along both the time dimension and the cross-section dimension for e_{it}, without affecting the main properties of factor estimates. Such weak correlations in errors give rise to what Chamberlain & Rothschild (1983) called the approximate factor structure.

set I_{t-1} with the feasible information vector $\hat{I}_{t-1} = (Y'_{t-1}, \hat{F}'_{t-1}) \in \Re^d$, $d = (p+1)+k$, where $\hat{F}_{t-1} = (\hat{F}_{1,t-1}, ..., \hat{F}_{k,t-1}) \in \Re^k, k \in \aleph$, is the vector of estimated factors from the panel data $X_{i,t-1}$. In the remainder of this paper we assume that the time series process $\{(Y_t, F'_{t-1})' : t = 0, \pm 1, \pm 2, ...\}$, defined on the probability space $(\Omega, \mathcal{A})$ is strictly stationary and ergodic.

Under the assumption that the conditional distribution of Y_t given I_{t-1} is continuous, we can then define the τ^{th} conditional quantile of Y_t given I_{t-1} as the measurable function q_τ satisfying the conditional restriction

$$P(y_t \leq q_\tau(I_{t-1}) \mid I_{t-1}) = \tau, \text{ almost surely.} \tag{2}$$

We use the Quantile Autoregression (QAR) model of order p as that is set out in Koenker & Xiao (2006) so that the variable of interest can then be characterised by the following equation

$$y_t = \theta_0(u_t) + \theta_1(u_t) y_{t-1} + ... + \theta_p(u_t) y_{t-p}, \tag{3}$$

where u_t is a sequence of i.i.d. standard uniform random variable, while $\theta_i(u_t)$ are unknown functions $[0,1] \rightarrow \Re$. Provided that the right hand side of (3) is monotone increasing in u_t, the τ^{th} conditional quantile function of y_t takes the following form,

$$\begin{aligned} m(Y_{t-1}, \theta(\tau)) &= \theta_0(\tau) + \theta_1(\tau) y_{t-1} + ... + \theta_p(\tau) y_{t-p} \\ &= Y_{t-1}\theta(\tau). \end{aligned} \tag{4}$$

Our objective is to test whether factors are an omitted variable in a quantile autoregressive model for y_t, while at the same time controlling for dynamic misspecification. In the sequel, we are testing the following hypothesis:

$$H_{0,1}: \quad E[\mathbb{1}(y_t - Y_{t-1}\theta(\tau)) \leq 0) - \tau | I_{t-1}] = 0 \qquad \text{a.s. for some } \theta \in \mathcal{B} \text{ and for all } \tau \in \mathcal{T} \tag{5}$$

against its respective alternative

$$H_{A,1}: \quad Pr\{E[\mathbb{1}(y_t - Y_{t-1}\theta(\tau)) \leq 0) - \tau | I_{t-1}] = 0\} < 1, \tag{6}$$

where $\mathcal{B}$ is a family of uniformly bounded functions from $\mathcal{T}$ to Θ.

The null hypothesis states that if the specification is correct, the probability that the observed value of y_t falls below the estimated quantile should, on average, equal the nominal quantile level of interest

(τ) with probability one. Hypothesis $H_{0,1}$ is the joint hypothesis that the QAR model is not dynamically misspecified and factors are not an omitted variable. If $H_{0,1}$ did not include such latent factors, the null hypothesis would simplify to the null hypothesis of the correct specification of parametric dynamic quantile models, tested in Escanciano & Velasco (2010), where the conditioning information vector includes only directly observable variables.

Testing for the null hypotheses $H_{0,1}$ is a challenging problem, since it involves an infinite number of conditional moments indexed by $\tau \in \mathcal{T}$, where $\mathcal{T}$ is a compact set comprising of the range of quantiles of interest ($\mathcal{T} \subset (0,1)$). Therefore, following Bierens (1990) we can characterise $H_{0,1}$ by the infinite number of unconditional moment restrictions:

$$E\{[\mathbb{1}(y_t - Y_{t-1}\theta(\tau)) \leq 0) - \tau]exp(\xi'\mathcal{M}(I_{t-1}))\} = 0 \tag{7}$$

where $\mathcal{M}$ is an arbitrary Borel Measurable bounded one-to-one mapping from $\Re^d$ to $\Re^d$ and $\xi \in \Re^d$ is a vector of weights. Conditioning on I_{t-1} is equivalent to conditioning on the bounded vector $\mathcal{M}(I_{t-1})$, for I_{t-1} and $\mathcal{M}(I_{t-1})$ generate the same Borel field. Furthermore, we wish for the weight attached to past observations to decrease over time, thus we define the weighting vector ξ as in de Jong (1996). Let therefore, $exp(\xi'\mathcal{M}(Z_t)) = exp(\sum_{j=1}^{t-1}\xi_j'\mathcal{M}(Z_{t-j}))$ and $\Xi = \{\xi_j : a_j \leq \xi_j \leq \gamma_j, j = 1,2; |a_j|, |\gamma_j| \leq \Gamma j^{-\kappa}, \kappa \geq 2\}$.

Given therefore a sample $\{(y_t, \hat{I}_{t-1}')' : 1 \leq t \leq T\}$ and an estimated parameter value $\hat{\theta}(\tau)$, we consider the following quantile empirical process:

$$S_{1,T}(\xi, \hat{\theta}_T(\tau)) := T^{-\frac{1}{2}} \sum_{t=1}^{T} [\mathbb{1}(y_t - Y_{t-1}\hat{\theta}_T(\tau) \leq 0) - \tau]exp(\xi'\mathcal{M}(\hat{I}_{t-1})) \tag{8}$$

for the Quantile Autoregression Estimator (QARE), proposed by a Koenker & Xiao (2006)), defined as

$$\hat{\theta}_T(\tau) = \arg\min_{\theta \in \mathbb{R}^{p+1}} \sum_{t=1}^{T} \rho_\tau(y_t - Y_{t-1}\theta) \tag{9}$$

where $\rho_\tau(u) = u(\tau - \mathbb{1}(u < 0))$ is the "tick" loss function.

The null hypothesis holds when the process $S_{1,T}(\xi, \hat{\theta}_T(\tau))$ is close to zero for almost all $(\xi', \tau)' \in \Re^d \times \mathcal{T}$, and thus the test statistic is based on a distance from a standardised sample analogue of $E\{[\mathbb{1}(y_t - Y_{t-1}\hat{\theta}_T(\tau) \leq 0) - \tau]exp(\xi'\mathcal{M}(\hat{I}_{t-1}))\}$ to zero. Some popular norms we could consider are the following:

- Cramer-von-Mises $\Rightarrow CvM_{j,T} := \int_{\mathcal{T}} \int_{\mathcal{Y}} |S_{1,T}(\xi, \hat{\theta}_T(\tau))|^2 d\Phi_1(\xi) d\Phi_2(\tau)$

- Kolmogorov-Smirnov $\Rightarrow KS_{j,T} := sup_{\tau \in \mathcal{T}} \int_{\mathcal{Y}} |S_{1,T}(\xi, \hat{\theta}_T(\tau))|^2 d\Phi_1(\xi)$

where Φ_1 and Φ_2 are some integrating measures on $\mathcal{Y}$ and $\mathcal{T}$ respectively and $\mathcal{Y}$ is a generic compact subset of $\Re^d$ containing the origin.

The test we propose therefore rejects H_0 for "large" values of such functionals. If $H_{0,1}$ is not rejected then one can conclude that there is no evidence that the QAR model is misspecified with omitted variables and thus can proceed with inference. If, however, $H_{0,1}$ is rejected we still need to ascertain the source of the rejection. A logical next step would be to augment the quantile autoregression model with the feasible estimated factors. We therefore proceed to test the following null hypothesis:

$$H_{0,2}: \quad E[\mathbb{1}(y_t - Y_{t-1}\theta_1(\tau) - F_{t-1}\theta_2(\tau) \leq 0) - \tau | I_{t-1}] = 0 \quad \text{a.s. for some } \theta \in \mathcal{B} \text{ and for all } \tau \in \mathcal{T} \tag{10}$$

against its negation

$$H_{A,2}: \quad Pr\{E[\mathbb{1}(y_t - Y_{t-1}\theta_1(\tau) - F_{t-1}\theta_2(\tau) \leq 0) - \tau | I_{t-1}] = 0\} < 1 \tag{11}$$

In this case, the null hypothesis $H_{0,2}$ states that there is no evidence that the Factor-augmented QAR (FA-QAR) is not correctly specified. It follows that the test statistic of interest for $H_{0,2}$ will be based on the quantile empirical process $S_{2,T}(\xi, \hat{\theta}_T(\tau)) := T^{-\frac{1}{2}} \sum_{t=1}^{T} [\mathbb{1}(y_t - Y_{t-1}\hat{\theta}_{1,T}(\tau) - \hat{F}_{t-1}\hat{\theta}_{2,T}(\tau) \leq 0) - \tau] exp(\xi' \mathcal{M}(\hat{I}_{t-1}))$. In this instance the factor estimation error not only appears in the conditioning information vector, but also influences the indicator function element of the statistic. If the null hypothesis fails to reject then there is no evidence that the FAQAR is misspecified, hence factors were originally an omitted variable. If on the other hand, $H_{0,2}$ is also rejected then we have evidence that the linear specification of the model may not be appropriate and alternative specifications might need to be taken into consideration - however, this is beyond the scope of this paper.

2.2 Assumptions

Let $\|A\| = [tr(A'A)^{\frac{1}{2}}]$ denote the norm of matrix A. Throughout, we let F_t be the $k \times 1$ vector of true factors and λ_i be the true loadings, with F and Λ being the corresponding matrices. The relevant assumptions for the latent factors are those used in Bai (2003) to derive the limiting distributions of the estimated factors, factor loadings and common components and are provided explicitly in the Appendix as Assumptions A-F. We rely on those same assumptions, to demonstrate that factor estimation error does not influence our test statistic.

To derive the asymptotic results of the quantile empirical process, we need to further consider the following assumptions. Let, for each $t \in \mathbb{Z}$, $\mathcal{F}_t = \sigma(I_t', I_{t-1}', \ldots)$ be the σ-field generated by the information set obtained up to time t. Define also the family of conditional distributions $F_b(y) := P(Y_t \leq y | I_{t-1} = b)$. Let f_b be the density function of the cumulative distribution function (cdf) F_b. In particular, $f_{I_{t-1}}(y)$ denotes the density of Y_t given I_{t-1}, evaluated at y. Also, the family $\mathcal{B}$, in which the parameter θ takes values, is endowed with the sup norm (i.e. $\|\theta\|_{\mathcal{B}} = \sup_{\tau \in \mathcal{T}} |\theta(\tau)|$). Lastly, given that similar assumptions are needed both when testing the QAR model under $H_{0,1}$ and FA-QAR model under $H_{0,2}$, let $m_j(I_{t-1}, \theta(\tau))$ be the conditional quantile function under consideration and $H_{0,j}$ the null hypothesis to be tested. Thus, $m_1(I_{t-1}, \theta(\tau)) = Y_{t-1}\theta(\tau)$ is the conditional quantile function in the QAR case and $m_2(I_{t-1}, \theta(\tau)) = Y_{t-1}\theta_1(\tau) + F_{t-1}\theta_2(\tau)$ in the FA-QAR case.

Assumption G: Time series Model Checks

1. $\{(Y_t, F_t')' : t = 0, \pm 1, \pm 2, \ldots\}$ is a strictly stationary and ergodic process. Under $H_{0,j}$, $\{\mathbb{1}(y_t - m_j(I_{t-1}, \theta(\tau)) \leq 0) - \tau, \mathcal{F}_t\}$ is a martingale difference sequence for all $\tau \in \mathcal{T}$.

2. $m_j(I_{t-1}, \theta(\tau))$ is non-decreasing in τ a.s.

3. The family of distributions functions $\{F_b, b \in \Re^d\}$ has Lebesgue measures $\{f_b, b \in \Re^d\}$ that are uniformly bounded away from zero for the quantiles of interest.

Assumption H: Class of functions

For each general $\theta_1 \in \mathcal{B}$,

1. There exists a vector of functions $g_{t-1} : \Theta \to \Re^q$ such that $g_{t-1}(\theta_1(\tau))$ is $\mathcal{F}_{t-1}$-measurable for each $t \in \mathcal{Z}$, and satisfies for all $k < \infty$,

$$\sup_{1 \leq t \leq n, \|\theta_1 - \theta_2\|_{\mathcal{B} \leq kT^{-\frac{1}{2}}}} T^{\frac{1}{2}} \|m_j(I_{t-1}, \theta_2) - m_j(I_{t-1}, \theta_1) - (\theta_2 - \theta_1)' g_{t-1}(\theta_1)\|_{\mathcal{B}} = o_p(1).$$

2. For all sufficiently small $\delta > 0$,

$$E\Big[\sup_{\|\theta_1 - \theta_2\|_{\mathcal{B} \leq \delta}} |\mathbb{1}(y_t - m_j(I_{t-1}, \theta_1(\tau)) \leq 0) - \mathbb{1}(y_t - m_j(I_{t-1}, \theta_2(\tau)) \leq 0)|\Big] \leq C\delta, \quad \text{for all } \tau \in \mathcal{T}, \text{ and}$$

$$E\Big[\sup_{|\tau_1 - \tau_2| \leq \delta} |m_j(I_{t-1}, \theta_1(\tau_1)) - m_j(I_{t-1}, \theta_1(\tau_2))|\Big] \leq C\delta.$$

3. Uniformly in $\tau \in \mathcal{T}$, $E|g_{t-1}(\theta_1(\tau))|^2 < \infty$, and uniformly in $(\xi, \tau) \in \mathcal{Y} \times \mathcal{T}$,

$$|T^{-1}\sum_{t=1}^{T} g_{t-1}(\theta(\tau))exp(\xi'\mathcal{M}(I_{t-1}))f_{I_{t-1}}(m_j(I_{t-1},\theta_0)) - E[g_{t-1}(\theta(\tau))exp(\xi'\mathcal{M}(I_{t-1}))f_{I_{t-1}}(m_j(I_{t-1},\theta_0))]| = o_p(1$$

Assumption I: Compactness of the parameter space

The parametric space Θ is compact in $\Re^p$. The true parameter $\theta(\tau)$ belongs to the interior of Θ for each $\tau \in \mathcal{T}$ and $\theta \in \mathcal{B}$. The class $\mathcal{B}$ satisfies

$$\int_0^\infty \left(log(N_{[\cdot]}\delta^2, \mathcal{B}, \|\cdot\|_{\mathcal{B}})\right)^{\frac{1}{2}} d\delta < \infty.$$

Assumption J: Estimator Consistency

The estimator $\hat{\theta}_T$ satisfies that $P(\hat{\theta}_T \in \mathcal{B}) \to 1$ as $T \to \infty$, and the following asymptotic expansion under H_0,

$$Q_T(\tau) = \sqrt{T}(\hat{\theta}_T(\tau) - \theta(\tau))$$

$$= \tfrac{M^{-1}}{q(\tau)} T^{-\frac{1}{2}} \textstyle\sum_{t=1}^{T} (\mathbb{1}(y_t - m_j(I_{t-1}, \theta(\tau)) \leq 0) - \tau) I_{t-1} + o_p(1), \quad \text{uniformly in } \tau \in \mathcal{T},$$

where $M = E(\mathbf{I'I})$ is a positive definite matrix, and $q(\tau) = f_\epsilon(F_\epsilon^{-1}(\tau))$ is the reciprocal of the sparsity function. Furthermore, the process $Q_T(\tau)$ converges weakly to a zero mean Gaussian process $Q(\tau)$ with covariance function

$$Cov_Q(\tau_1, \tau_2) = \frac{M^{-1}}{q(\tau_1)} \times L(\theta(\tau_1), \theta(\tau_2)) \times \frac{M^{-1}}{q(\tau_2)} \quad (12)$$

where

$$L(\theta(\tau_1), \theta(\tau_2)) = \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T}\sum_{s=1}^{T} E\Big[(\mathbb{1}(y_t - m_j(I_{t-1}, \theta(\tau_1)) \leq 0) - \tau_1) I_{t-1} \times (\mathbb{1}(y_s - m_j(I_{s-1}, \theta(\tau_2)) \leq 0) - \tau_2) I_{s-1}\Big]. \quad (13)$$

Assumption G1 is standard in time series model checks and is always true in the present context where the information set contains all relevant past history, while G3 is necessary for the asymptotic tightness of the process $S_{j,T}(\xi, \theta_T)$. Assumption H is satisfied for the Linear Quantile Autoregression model under consideration. Sufficient conditions for Assumption I of monotone classes of functions applying to the QAR model can be found in Theorem 2.7.5 in Vaart & Wellner (2000). Meanwhile, the asymptotic normality of the quantile regression process has been established in the literature under a variety of conditions, see for example Theorem 1 in Gutenbrunner & Jureckova (1992).

2.3 Asymptotic Null Distribution

In this subsection we establish the limit distribution of the quantile-marked empirical process $S_{j,T}(\xi, \hat{\theta}_T(\tau))$ under the null hypothesis $H_{0,j}$.

We begin by showing that the factor estimation error in the feasible information vector is negligible and therefore the proposed statistics converge to the equivalent statistics with the infeasible information vector that includes the true latent factors and a term which converges to zero.

Lemma 1: Let Assumptions A-F (shown in Appendix) hold. Then:

$$\begin{aligned} S_{j,T}(\xi, \hat{\theta}_T(\tau)) &:= T^{-\frac{1}{2}} \sum_{t=1}^{T} [\mathbb{1}(y_t - m_j(\hat{I}_{t-1}, \hat{\theta}(\tau)) \leq 0) - \tau] exp(\xi' \mathcal{M}(\hat{I}_{t-1})) \\ &= T^{-\frac{1}{2}} \sum_{t=1}^{T} [\mathbb{1}(y_t - m_j(I_{t-1}, \hat{\theta}(\tau)) \leq 0) - \tau] exp(\xi' \mathcal{M}(I_{t-1})) + o_p(1) \end{aligned} \quad (14)$$

It is immediate to see that in a first instance, when dealing with $S_{1,T}(\xi, \hat{\theta}_T(\tau))$, factor estimation error is only present in the conditioning set and thus only appears in the weighting exponential function, while in the test statistic $S_{2,T}(\xi, \hat{\theta}_T(\tau))$, the factor estimation error is present in both the exponential function and in the indicator function element. Therefore, the proof of the statement for $H_{0,1}$ follows from the proof of statement for $H_{0,2}$ shown in the Appendix.

We then proceed to state the limiting distribution of the test statistic, recognising that in addition to factor estimation error, which has been found to be negligible, there is also parameter estimation error present. Define the function $G(\xi, \theta(\tau)) = E[g_{t-1}(\theta(\tau)) f_{I_{t-1}}(m_j(I_{t-1}, \theta(\tau)) exp(\xi' \mathcal{M}(I_{t-1}))]$, $\xi \in \Upsilon$, $\tau \in \mathcal{T}$. Also, note that under a suitable central limit theorem, $S_{j,T}(\xi, \theta_T(\tau))$ converges to a zero mean Gaussian process $S_{j,\infty}(\xi, \theta(\tau))$ with covariance function given by $Cov_\infty(\nu_1, \nu_2) =$

$(\min\{\tau_1, \tau_2\} - \tau_1\tau_2)E\big[exp((\xi_1 - \xi_2)'\mathcal{M}(I_0))\big]$.

Theorem 1: Let Assumptions A-J hold. Then, under the null hypothesis $H_{0,j}$, $S_{j,T}(\xi, \hat{\theta}_T(\tau)) \xrightarrow{d} \sup_{\xi\in\Upsilon,\tau\in\mathcal{T}} |S_j(\xi, \theta(\tau))|$, where $S_j(\xi, \theta(\tau))$ is a zero mean Gaussian process with covariance function

$$\begin{aligned} Cov(\nu_1, \nu_2) = Cov_\infty(\nu_1, \nu_2) + G'(\xi_1, \theta(\tau_1))Cov_Q(\tau_1, \tau_2)G(\xi_2, \theta(\tau_2)) \\ - E\big[S_{j,\infty}(\xi_1, \theta(\tau_1))G'(\xi_2, \theta(\tau_2))Q(\tau_2)\big] - E\big[G'(\xi_1, \tau_1)Q(\tau_1)S_{j,\infty}(\xi_2, \theta(\tau_2))\big] \end{aligned} \quad (15)$$

where, $\nu_1 = (\xi_1', \tau_1)'$, $\nu_2 = (\xi_2', \tau_2)'$.

Details regarding the convergence of the statistic can be found in the Appendix, while the consistency properties of tests based on continuous functionals has been shown in Escanciano & Velasco (2010).

2.4 Critical Values Construction

Under the null hypothesis, the quantile error $S_{j,T}(\xi, \theta(\tau))$ converges to a Gaussian process with zero mean and a given covariance structure. However, when the estimated parameter $\hat{\theta}_T$ is used in $S_T(\xi, \hat{\theta}_T)$, the parameter estimation error affects its asymptotic properties. Given that the asymptotic null distribution of $S_{j,T}(\xi, \hat{\theta}_T)$ will be dependent on the data generating process and thus is not nuisance parameter free, critical values for the test statistics cannot be tabulated for general cases.

Bootstrap methods have been proposed in the literature for quantile regression models (see, e.g., Hahn (1995); Parzen *et al.* (1994)). With respect to quantile regression models with time series, Gregory *et al.* (2018) established a smooth tapered block bootstrap procedure. Their work studied the properties of the block bootstrap with smoothing of both data observations via kernel smoothing techniques and data blocks by tapering, which demonstrated the validity of the block bootstrap in dynamic quantile models. At the same time their work extended the validity of the moving block bootstrap (Fitzenberger, 1998) to quantile regression under weaker conditions than previously considered.

In our context, under both the null hypotheses, $[\mathbb{1}(y_t - m_j(I_{t-1}, \theta(\tau) \leq 0) - \tau]exp(\xi'\mathcal{M}(\hat{I}_{t-1})$ is a martingale difference sequence, therefore resampling blocks of length one, as in the iid case preserves the first order validity of the block bootstrap (Corradi *et al.*, 2009). In order to achieve higher order refinements, the block bootstrap with an increasing block size would have been necessary. However, given that our statistics depend on the nuisance parameters ξ that are not identified under the null, we cannot obtain such refinements. Therefore, in order to preserve the temporal ordering, we proceed to jointly resample $(y_t, Y_{t-1}, \hat{F}_{t-1}, exp(\sum_{j=1}^{t-1} \xi_j' \mathcal{M}(I_{t-j})))$ by drawing $T-1$ independent draws. For each bootstrap replication we use the same set of resampled values across $\xi \in \Xi$.

The bootstrap analogues of $S_{1,T}(\xi, \hat{\theta}_T(\tau))$ and $S_{2,T}(\xi, \hat{\theta}_T(\tau))$, say $S_{1,T}^*(\xi, \hat{\theta}_T^*(\tau))$ and $S_{2,T}^*(\xi, \hat{\theta}_T^*(\tau))$ respectively, are then defined to be

$$S_{1,T}^*(\xi, \hat{\theta}_T^*(\tau)) := T^{-\frac{1}{2}} \sum_{t=1}^{T} \Big[\mathbb{1}(y_t^* - Y_{t-1}^* \hat{\theta}_T^*(\tau) \leq 0) - \tau] exp(\xi' \mathcal{M}(\hat{I}_{t-1}^*)) \qquad (16)$$

$$- \mathbb{1}(y_t - Y_{t-1} \hat{\theta}_T(\tau) \leq 0) - \tau] exp(\xi' \mathcal{M}(\hat{I}_{t-1}))\Big]$$

$$S_{2,T}^*(\xi, \hat{\theta}_T^*(\tau)) := T^{-\frac{1}{2}} \sum_{t=1}^{T} [\mathbb{1}(y_t^* - Y_{t-1}^* \hat{\theta}_{1,T}^*(\tau) - \hat{F}_{t-1}^* \hat{\theta}_{2,T}^*(\tau) \leq 0) - \tau] exp(\xi' \mathcal{M}(\hat{I}_{t-1}^*)) \qquad (17)$$

$$- \mathbb{1}(y_t - Y_{t-1} \hat{\theta}_{1,T}(\tau) - \hat{F}_{t-1} \hat{\theta}_{2,T}(\tau) \leq 0) - \tau] exp(\xi' \mathcal{M}(\hat{I}_{t-1}))\Big].$$

Consequently, one could then obtain the corresponding bootstrap functionals

- $CvM_{j,T}^* := \int_{\mathcal{T}} \int_{\mathcal{Y}} |S_{j,T}^*(\xi, \hat{\theta}_T^*(\tau))|^2 d\Phi_1(\xi) d\Phi_2(\tau)$
- $KS_{j,T}^* := sup_{\tau \in \mathcal{T}} \int_{\mathcal{Y}} |S_{j,T}^*(\xi, \hat{\theta}_T^*(\tau))|^2 d\Phi_1(\xi)$.

For any bootstrap replication, we compute the bootstrap functional $CvM_{j,T}^*(KS_{j,T}^*)$. Performing then B bootstrap replications, with B large, we compute the quantiles of the empirical distribution of the B bootstrap statistics. The null hypothesis $H_{0,j}$ is rejected if $CvM_{j,T}(KS_{j,T})$ based on the original sample is greater than the $(1-\alpha)^{th}$ percentile of the corresponding bootstrap distribution, where α is the level of significance. This is because $CvM_{j,T}(KS_{j,T})$ has the same limiting distributions as its corresponding bootstrapped statistics, which ensures an asymptotic size equal to α. Under the alternative, $CvM_{j,T}(KS_{j,T})$ diverges to infinity, while the corresponding bootstrap statistics maintain its well defined distribution, ensuring asymptotic power.

3 Finite Sample Performance

We consider two simulation cases to assess the finite sample performance of the proposed test statistics. The data generating process follows the following autoregressive process:

$$y_t = \rho_0(\upsilon_t) + \rho_1(\upsilon_t)y_{t-1} + \cdots + \rho_p(\upsilon_t)y_{t-p} + \beta_1 F_{1,t-1} + \cdot + \beta_k(\upsilon_t)F_{k,t-1}$$

where υ_t are independent Uniform $(0,1)$ random variables . Motivated by our subsequent empirical application, the lag is taken as $p = 1$ and the varying coefficients are defined as follows:

$$\text{Case 1: } \rho_0(\upsilon) = 10 + \Phi^{-1}(\upsilon), \rho_1(\upsilon) = 0.5$$
$$\text{Case 2: } \rho_0(\upsilon) = 10 + \Phi^{-1}(\upsilon), \rho_1(\upsilon) = 0.5, \beta_1(\epsilon) = 0.8$$

where the intercept coefficient $\rho_0(\upsilon)$ is from a normal distribution, while the slope coefficients are constant.

As it is evident, under Case 1 the null hypothesis $H_{0,1}$ is satisfied, while under Case 2 the null hypothesis $H_{0,1}$ is violated, but $H_{0,2}$ is satisfied. We consider multiple sample sizes, including $T = 100$, $T = 300$, $T = 500$ and $T = 1000$, carry out 1000 Monte Carlo Simulations, and for each of them perform 300 bootstrap replications. In all the replications the nominal probability of rejecting a correct null hypothesis is 0.05. The conclusions with other nominal values are similar.

Following the relevant literature, we choose the exponential function instead of the indicator function, as it has been confirmed that exponential-based tests have higher performance than indicator-based tests. Furthermore, following the earlier relevant consistent test literature (see, e.g. Bierens (1982), Bierens (1990)), we choose $\mathcal{M}$ as the *arctan* function. In the experiment, we consider a grid of $\mathcal{T}$ in $m = 17$ equidistributed points from $\omega = 0.1$ to $1 - \omega = 0.9$. Denote by $\mathcal{T}_m = \{\tau_q\}_{q=1}^m$ the point in the grid, with $w = \tau_1 < \cdots < \tau_m = 1 - \omega$. Also, as already mentioned we wish to weigh more heavily the more recent lags. Let $\Psi_\tau(\epsilon_t) = \mathbb{1}(y_t - \rho_0(\tau) - \rho_1(\tau)y_{t-1} \leq 0) - \tau$ and let $exp(\xi_Z'\mathcal{M}(Z_t)) = exp(\sum_{j=1}^d \xi_j'\mathcal{M}(Z_{t-j}))$, where $d = d(t) = \min\{t-1, c\}$ with $c < \infty$ and $\Xi = \xi_j : a_j \leq \xi_j \leq \gamma_j, j = 1, 2; |a_j|, |\gamma_j| \leq \Gamma j^{-\kappa}, \kappa \geq 2$. It is immediate to see that the weight attached to past observations decreases over time. We define Γ over a grid, $\Gamma \in [0, 3]$ and evaluate $g = 30$ equidistributed points along the grid. Therefore the CvM and KS statistics are computed as:

$$KS_T = sup_{\tau \in \mathcal{T}} sup_{g \in \Gamma} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \Psi_q(\epsilon_t) exp\Big(\sum_{j=0}^{d} \xi_j'(tan^{-1}(Y_{t-1}) + tan^{-1}(\hat{F}_{t-1}))\Big) \tag{18}$$

$$CvM_T = \frac{1}{mg^2\sqrt{T}} \sum_{q=1}^{m} \sum_{b=1}^{g} \sum_{t=1}^{T} \Psi_q(\epsilon_t) exp\Big(\sum_{j=0}^{d} \xi_j' (tan^{-1}(Y_{t-1}) + tan^{-1}(\hat{F}_{t-1}))\Big) \tag{19}$$

The theory allows for $m \to \infty$ as $T \to \infty$ and the $\{\tau_q\}_{q=1}^m$ generated independently from a distribution on $\mathcal{T}$, but for simplicity in the computations we assume m fixed and $\{\tau_q\}_{q=1}^m$ deterministic throughout the remainder of this paper.

	T	Case 1		Case 2	
		KS_T	CvM_T	KS_T	CvM_T
$\rho_1 = 0.5, \beta_1 = 0.8$	100	0.000	0.034	0.056	0.074
	300	0.000	0.032	0.438	0.968
	500	0.002	0.028	0.706	1.000
	1000	0.006	0.036	0.950	1.000

Table 1: Empirical size and power (rejection frequencies)

In Table 1 we report the rejection frequencies for the test of the null hypothesis $H_{0,1}$ based on the two continuous functionals, the *Kolmogorov-Smirnov* and the *Cramer-von-Mises*, for the model under the two data generating processes.[3] The empirical size, though smaller than the nominal 5% is satisfactory for the CvM functional even with a low number of observations, in contrast with the KS functional which remains severely undersized. With regards to the empirical power, we observe that the CvM has the highest rejection frequencies, however both functionals capture the misspecification under the DGP of Case 2 relatively well, when the null hypothesis is that of a standard quantile autoregressive model with no additional factors (i.e. $H_{0,1}$). This is in line with previous empirical results which have shown that the CvM functional outperforms the KS in terms of power (Escanciano & Velasco, 2010). Furthermore, these rejection rates increase with both the sample size and the strength of the alternative hypothesis imposed (i.e. with higher values of β_1).

This limited simulation study suggests that even with relatively small sample sizes (a common scenario in empirical applications), the test exhibits fairly good size accuracy and power performance, particularly when using the CvM functional. As a result, in the subsequent empirical application, we will be basing our analysis only on the *Cramer-von-Mises* functional.[4]

[3] No simulations are provided here for the test of the null hypothesis $H_{0,2}$, as it is more related to other forms of misspecification, rather than omitted latent factors.

[4] It is worth noting that one could also employ the *Kupiec* functional.

4 Application To The Distributions of GDP Growth And CPI Inflation

GDP growth and CPI inflation are two of the most important variables due to their dominant role in many macroeconomic models (Levin & Piger, 2002; Angeloni *et al.*, 2006). In the empirical literature, the majority of the work has examined how univariate regressions can be improved by augmenting such models with factors as a way of summarising large amounts of information in estimates and forecasts of the conditional mean of these variables (Stock & Watson, 2002; Bernanke *et al.*, 2005). Nevertheless, such point forecasts ignore the risks around the central forecast. For example, in the case of growth a central forecast may paint an overly optimistic picture of the state of the economy, which is why the policymakers' focus on downside risk has increased in recent years (Adrian *et al.*, 2019). Similarly, as argued by, inter alia, Henry & Shields (2004), the dynamic behaviour of inflation in particular has a number of economic implications.

"Fan charts" have been extensively used by the Bank of England in its Inflation Reports, to describe its best prediction of future inflation to the general public since 1997. However, more recently, a number of inflation-targeting central banks have started to publish both GDP growth and inflation distributions. Examining the dynamics and higher moments of inflation and growth is of extreme importance and the ability to test whether these conditional quantile models should be augmented with latent factors representing a larger information set (e.g. financial conditions) is necessary.

4.1 Data

In this section, in an illustrative empirical analysis, we aim to examine whether the quantiles of GDP growth rate and CPI inflation rate are best modelled as univariate regressions or factor augmented models. For the estimation of factors, we consider non-seasonally adjusted series containing data on inflation, real activity and indicators of money and key asset prices for the United Kingdom. We will employ the specification test using a sample which includes quarterly data from 177 macroeconomic variables, including the annual CPI inflation and GDP growth rates, spanning from the second quarter of 1991 to the second quarter of 2018, with a total of $T = 109$ observations. According to the criteria outlined in Bai & Ng (2002), the number of common latent factors present in the data is one and for both of the dependent variables in consideration, the autoregressive model of order 1 has been deemed appropriate by the Schwarz criterion outlined in Machado (1993).

4.2 Specification Test

We entertain for both the GDP growth and the CPI inflation rates a Linear QAR model of order 1, as in (4), and a factor augmented QAR of order 1, with parameters estimated by quantile autoregression. In order to test $H_{0,1}: \quad E[\mathbb{1}(y_t - \theta_0(\tau) - y_{t-1}\theta_1(\tau)) \leq 0) - \tau | I_{t-1}] = 0$, a.s. versus $H_{A,1}: \quad Pr\{E[\mathbb{1}(y_t - \theta_0(\tau) - Y_{t-1}\theta_1(\tau)) \leq 0) - \tau | I_{t-1}] = 0\}$, we construct the $S_{1,T}$ statistics as defined in (8), setting $I_{t-1} = [Y_{t-j}, F_{1,t-j}]'$. Similarly, in order to test $H_{0,2}: \quad E[\mathbb{1}(y_t - \theta_0(\tau) - Y_{t-1}\theta_1(\tau) - F_{t-1}\theta_2(\tau)) - \leq 0) - \tau | I_{t-1}] = 0$, a.s. versus $H_{A,1}: \quad Pr\{E[\mathbb{1}(y_t - \theta_0(\tau) - y_{t-1}\theta_1(\tau) - f_{t-1}\theta_2(\tau)) \leq 0) - \tau | I_{t-1}] = 0\}$, we construct the $S_{2,T}$ statistic. We use the exponential function and set $\mathcal{M}$ as the inverse tangent function, while setting $\xi_j = \Gamma(j+1)^{-2}$, where Γ is defined over a fine grid, $\Gamma = \binom{\Gamma_1}{\Gamma_2} \in [0,3] \times [0,3]$. Similarly with the simulation setup, we take m equidistributed points $\{\tau_g\}_{g=1}^{m}$ from $\tau_1 = 0.1$ to $\tau_m = 0.9$, but perform the test for multiple choices of m. In Table 2 we report the p-values for the CvM statistic obtained with the iid bootstrap with the number of bootstrap replications set to 500.

	m	QAR	FA-QAR		m	QAR
Dependent Variable GDP growth	5	0.006	0.330	**Dependent Variable CPI Inflation**	5	0.388
	9	0.004	0.380		9	0.314
	17	0.002	0.438		17	0.342

Table 2: Bootstrapped p-values for the $CvM_{1,T}$ and $CvM_{2,T}$ functionals

From the results in Table 2, we can conclude that the $QAR(1)$ model of GDP growth is misspecified at 1% nominal level, conditional on an information set that includes growth lags and the estimated factor, for all the number of quantiles estimated. We see that the omnibus test based on the statistic $CvM_{1,T}$ strongly rejects this model and the rejection appears stronger when the grid over $\mathcal{T}$ becomes finer, a result that is consistent with the fact that the power of the test improves as $m \to \infty$. Our expectation that the factor augmented model should be an omitted variable in the QAR model is verified with the corresponding p-values, where we see that the evidence of misspecification is eliminated as we fail to reject the null hypothesis $H_{0,2}$ (the FA-QAR columns), which indicates that the factor-augmented QAR is not misspecified. On the other hand, we see that there is no evidence to suggest that $QAR(1)$ model of CPI Inflation is not correctly specified and, in fact, the estimated latent factor is not an omitted variable and does not carry relevant information. This result is consistent with the literature supporting that past inflation is the most important determinant and thus best predictor for future inflation.

Based on these results, we clearly see that estimated latent factors may often carry information that is relevant for the estimation of variables of interest and having a test which enables us to test if they are an omitted variable, is of critical importance in empirical analysis. We further complement the testing procedure by estimating multiple quantiles of GDP growth and CPI inflation, employing the Factor Augmented Quantile Autoregression and the Quantile Autoregression respectively, in an attempt to trace out their distributions. In addition we examine the impact of the independent variables of our regression on the dependent variable.

Figure 1 shows the scatter plots of one-quarter-ahead CPI inflation against the current inflation rate and one-quarter-ahead GDP growth against the current growth rate and the current latent factor, along with the corresponding univariate regressions for the tenth, fiftieth and ninetieth quantile and the least squares regression line. The slopes for the CPI inflation lag appear relatively stable across the quantiles and so do the slopes of the GDP growth lag. Interestingly, the slopes of the latent factor differ significantly across the quantiles, implying that its impact is distinctively different across the spectrum of the GDP growth distribution. Furthermore, we see that for all the variables concerned the mean and median regression appear identical, which one might consider as an indication of symmetry in the distribution of the dependent variables.

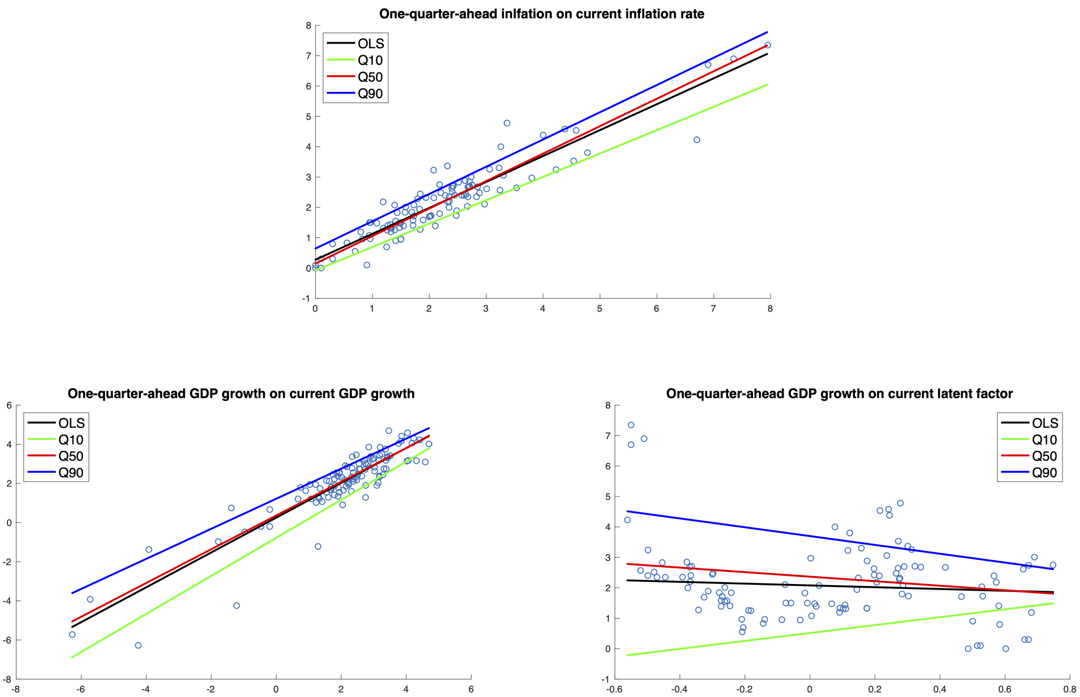

Figure 1: Univariate Quantile Regressions

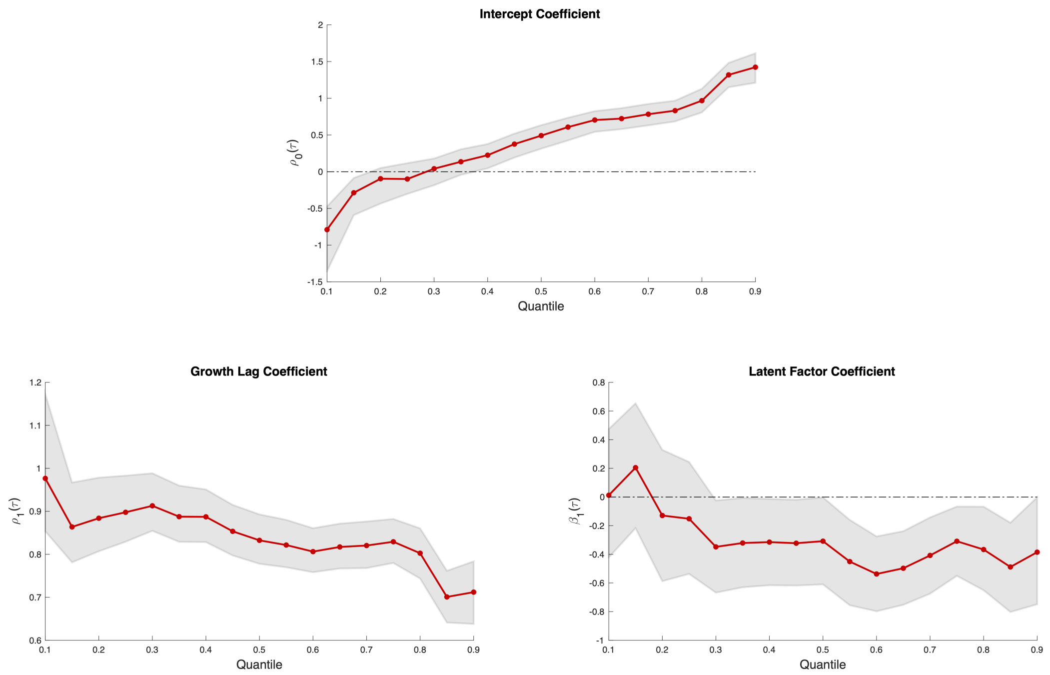

Figure 2: Coefficients for the FA-QAR(1,1) model of GDP growth with 90% confidence bands

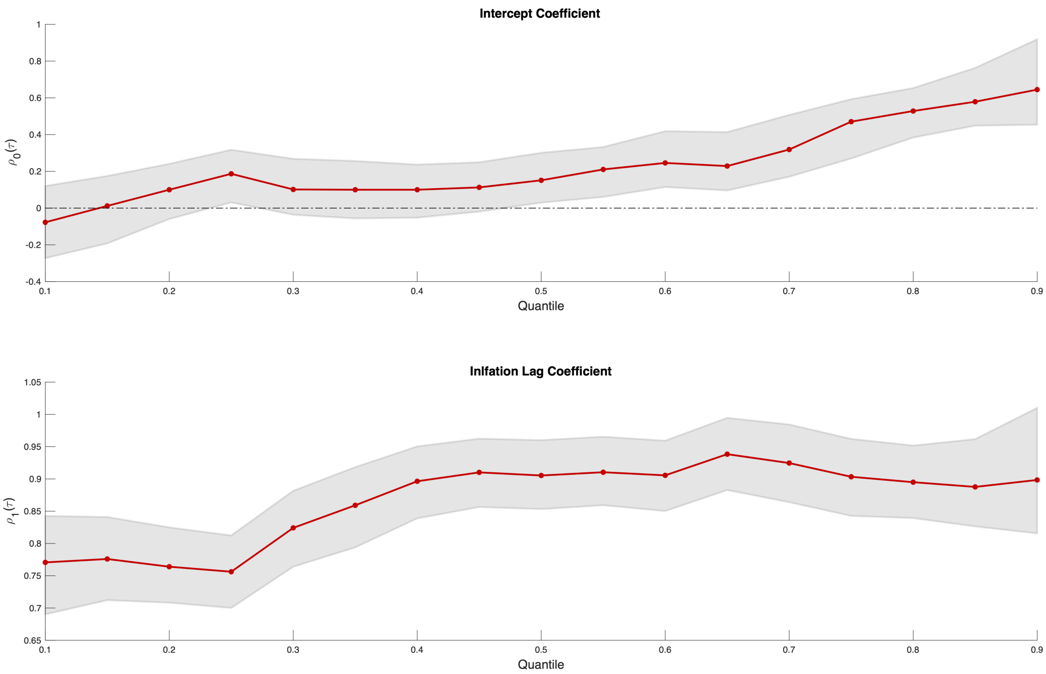

Figure 3: Coefficients for the QAR(1) model of CPI inflation with 90% confidence bands

Indeed, Figure 2 reveals that the autoregressive coefficient of GDP growth changes slightly across the evaluated quantiles and appears to become weaker across the upper tail of the distribution. This is not the case, however, for the latent factor coefficient. The current latent factor seems to be statistically not significant in the lower parts of the distribution, but its impact increases on the middle and towards the upper quantiles of the distribution, as the coefficient becomes more negative. This implies that the latent factor is not uniformly informative for predicting tail outcomes. For lower quantiles of GDP growth, the lag coefficient is the sole determinant, however, as we start to consider the upper tail of the distribution, the latent factor seems to provide significant information. In the case of CPI inflation, the autoregressive coefficient is statistically different from zero across all the estimated quantiles. The coefficient has a bigger impact on higher quantiles and in the extreme upper tail approaches a unit root process (see Figure 3).

Furthermore, in figure 4 we have traced out the tenth, fiftieth and ninetieth conditional quantiles across the sample time span. The figure illustrates three important empirical facts. On the one hand, if we focus on the periods before and around the 2008 economic crisis, we can see an asymmetry in the distribution of GDP growth as the difference between the ninetieth quantile and the median is significantly smaller than the difference between the median and tenth quantile. This asymmetry however seems to dissipate following the recent financial crisis up until the end of the sample. Furthermore, the distribution of GDP growth has remained fairly stable across the sample, with the exception of the 2008 recession, where GDP had experienced a significant decline. The distribution of CPI Inflation, on the other hand, has experienced smaller fluctuations throughout the sample and demonstrates a more symmetric behaviour. It is noticeable, however, that the CPI inflation distribution has narrowed significantly around 2015, when the interest rate was close to the zero lower bound.

Based on these indications, we finally proceed to smooth the estimated quantile distributions each quarter by interpolating between the estimated quantiles using the skewed $t-distribution$ by Azzalini & Capitanio (2003), characterised by four parameters that pin down the location, μ, scale, σ, fatness, ν, and shape, α. We fit the skewed $t-distribution$ in order to smooth the quantile function and recover a probability density function:

$$f(y;\mu,\sigma,\alpha,\nu) = \frac{2}{\sigma}t\Big(\frac{y-\mu}{\sigma};\nu\Big)T\Big(\alpha\frac{y-\mu}{\sigma}\sqrt{\frac{\nu+1}{\nu+(\frac{y-\mu}{\sigma})^2}};\nu+1\Big) \tag{20}$$

where $t(\cdot)$ and $T(\cdot)$, respectively, denote the PDF and CDF of the student $t-$distribution. For each quarter, we choose the four parameters $\{\mu,\sigma,\nu,\alpha\}$ of the skewed $t-distribution$ f to minimise the squared distance between our estimated quantile function $m_j(I_{t-1},\hat{\theta}(\tau))$ and the quantile function of the skewed $t-distribution$ $F^{-1}(\tau;\mu_t,\sigma_t,\nu_t,\alpha_t)$ to match the $5,25,75,95$ percent quantiles. This approach, first presented in Adrian *et al.* (2019), is computationally less burdensome while also making fewer parametric assumptions compared to alternative ways of estimating conditional predictive distributions (Hamilton, 1989; Smith & Vahey, 2016) and it allows us to obtain an estimated conditional distribution of the GDP growth and CPI inflation, plotted in Figures 5 & 6.

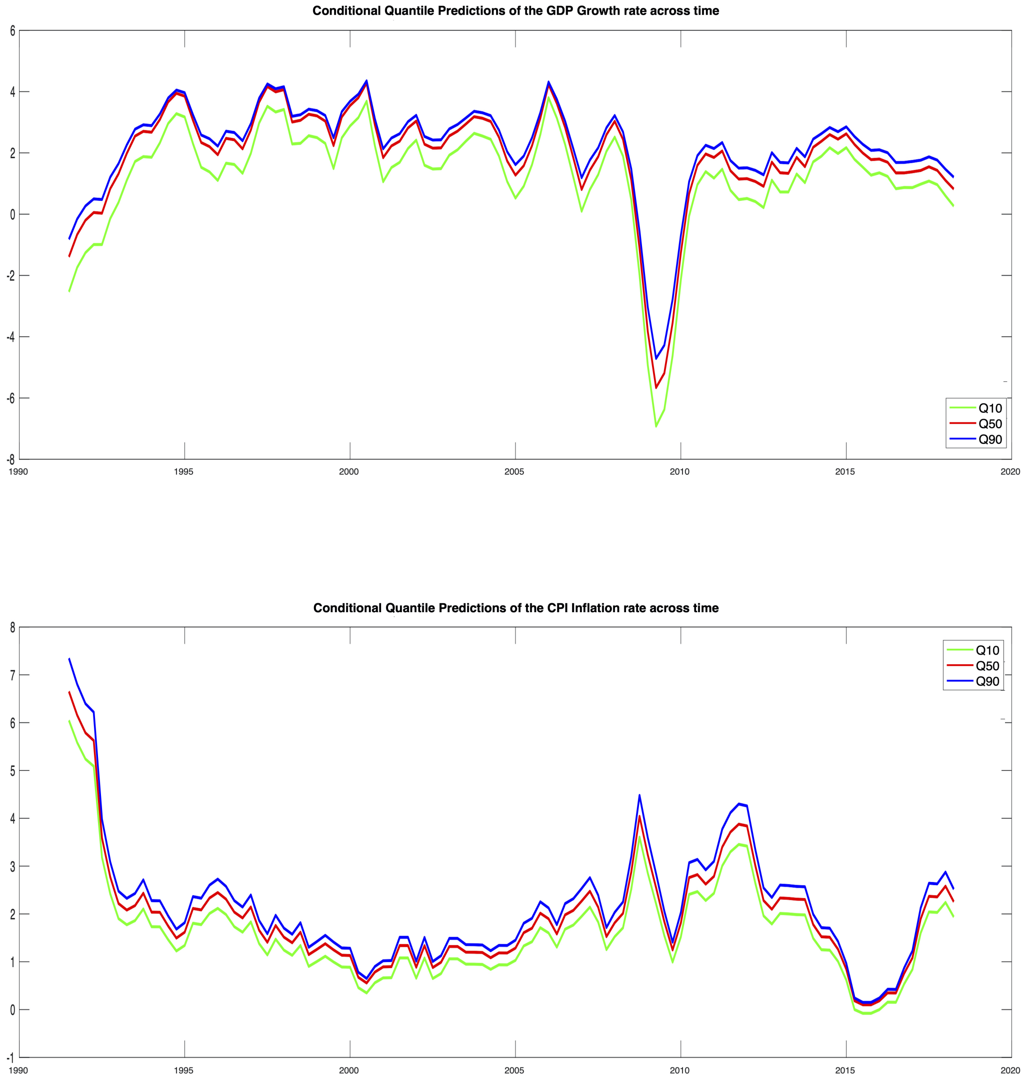

Figure 4: Estimated Conditional Quantiles

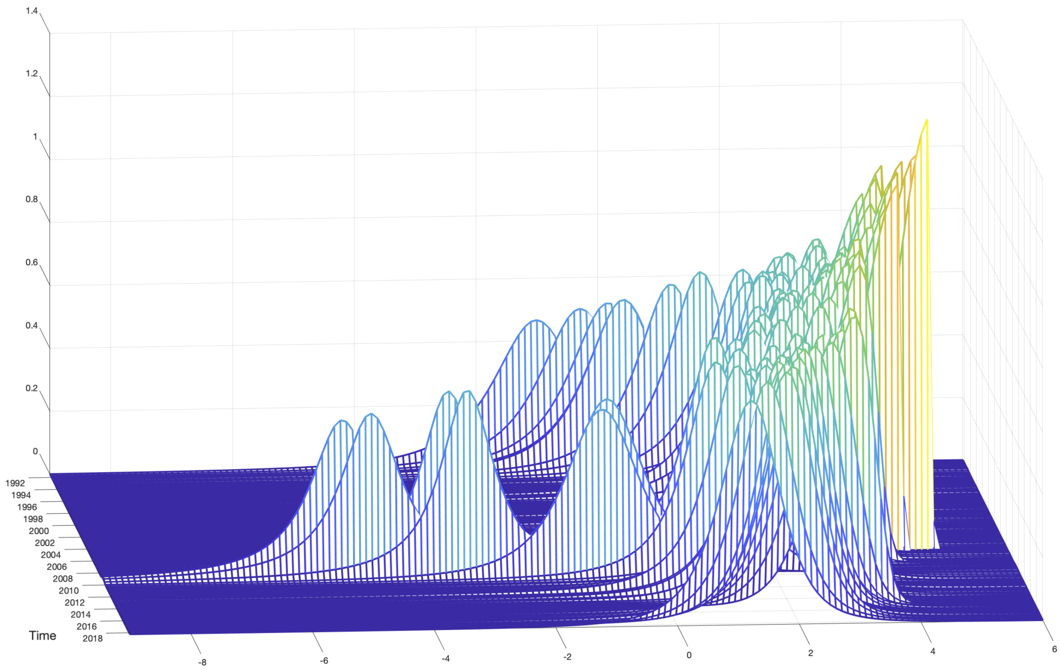

Figure 5: Distribution of GDP growth over time

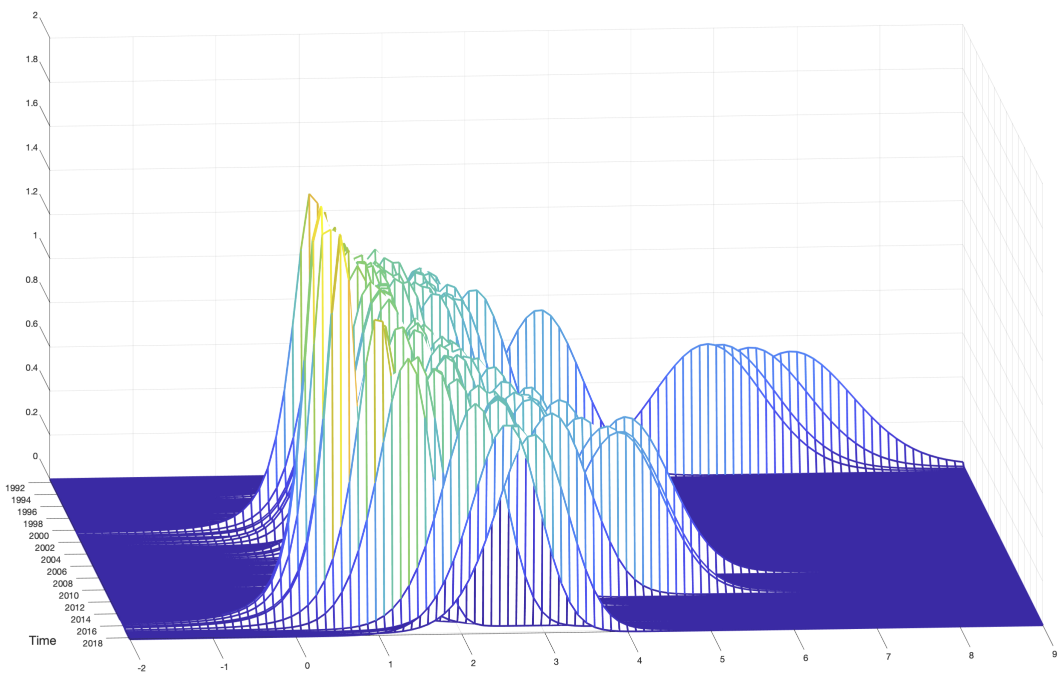

Figure 6: Distribution of CPI inflation over time

As it is evident in Figure 5, the entire distribution and not just the central tendency of the GDP growth has evolved over time. For example, the 2008 recession was associated with a rather symmetric distribution, albeit one with a significantly lower mean, while the antecedent expansion was associated with left skewed distributions. Furthermore, the right spectrum of the distribution appears to be more stable than the median and lower tails, which exhibit a stronger time variation. The distinctive behaviour in the tails of the conditional distribution indicates that when it comes to GDP, growth downside risk varies much more strongly than upside risk over time, a fact that has also been found to be true for the US by (Adrian *et al.*, 2019).

The distribution of CPI inflation on the other hand, with the exception of some outliers at the beginning of the sample, has remained fairly stable over time. The fluctuations in the central tendency have been significantly smaller than the fluctuations in the tails of the distribution, as well as the kurtosis. Furthermore, we see that the left tail of the distribution has remained fairly stable, a fact associated with the zero lower bound of nominal interest rates, while the median and the right tail in particular exhibit stronger time series variation. We see therefore that upside risk for CPI inflation varies much more strongly and was a key ingredient during the 2008 financial crisis and for the following years, where as we see the right tail of the distribution had significantly shifted to the right. This, in fact, supports the thought that, although inflation has on average remain unaltered during the crisis, the CPI inflation distribution was still affected by the volatility in the macro-economy in a multitude of ways, in terms of location, scale, fatness and shape. The significant variation in the distributions of both variables of interest illustrate the importance of examining higher moments and dynamics rather than simple point forecasts, which ignore the risks surrounding that central forecast.

5 Conclusion

Econometric modelling often requires the specification of conditional quantile models for a range of quantiles of the conditional distribution. However, such models may often rely on unobservable variables that require estimation prior to modelling. For the evaluation of quantile autoregressive models, we propose a test for the joint hypothesis of correct dynamic specification and no omitted latent variable with "valid" asymptotic critical values obtained via a bootstrap procedure based on the entire history of available information. We have demonstrated, through a simulation study, the consistency of the proposed test and its high power. An illustrative empirical implementation of the test suggests that the modelling of the conditional quantiles of UK GDP growth can be improved with the inclusion of estimated latent factors characterising the economy, while CPI inflation does not require such additional information. Furthermore, an empirical analysis of the conditional dependence structure of both variables illustrates that downside risk for GDP growth exhibits higher fluctuations, while in the case of inflation, upside risk becomes more relevant.

6 Appendix

6.1 Assumptions on Latent Factors

Assumption A: Factors
$E\|F_t\|^4 \leq M < \infty$ and $T^{-1}\sum_{t=1}^{T} F_t F_t' \xrightarrow{p} \Sigma_F$ for some $k \times k$ positive definite matrix Σ_F.

Assumption B: Factor loadings
$E\|\lambda_i\| \leq \bar{\lambda} < \infty$ and $\|\frac{\Lambda\Lambda'}{N} - \Sigma_\Lambda\| \to 0$ for some $k \times k$ positive definite matrix Σ_Λ.

Assumption C: Time and cross-section dependence and heteroscedasticity
There exists a positive constant $M < \infty$ such that for all N and T,

1. $E(e_{it}) = 0, E|e_{it}|^8 < M$;

2. $E(e_s' e_t/N) = E(N^{-1}\sum_{i=1}^{N} e_{is}e_{it}) = \gamma_N(s,t), |\gamma_N(s,s)| \leq M$ for all s, and

   $$T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T} |\gamma_N(s,t)| \leq M;$$

3. $E(e_{it}e_{jt}) = c_{ij,t}$ with $|c_{ij,t}| \leq |c_{ij}|$ for some c_{ij} and for all t. In addition,

   $$N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} |c_{ij}| \leq M;$$

4. $E(e_{it}e_{js}) = c_{ij,ts}$ and $(NT)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} |c_{ij,ts}| \leq M$;

5. For every (t,s), $E|N^{-\frac{1}{2}}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]|^4 \leq M$.

Assumption D: Weak dependence between factors and idiosyncratic errors

$$E\Big(\frac{1}{N}\sum_{i=1}^{N} \|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} F_t e_{it}\|^2\Big) \leq M.$$

Assumption E: Weak dependence
There exists $M < \infty$ such that for all T and N, and for every $t \leq T$ and every $i \leq N$,

1. $\sum_{s=1}^{T} |\gamma_N(s,t)| \leq M$

2. $\sum_{k=1}^{N} |c_{ki}| \leq M$

Assumption F: Moments and central limit theorem

There exists $M < \infty$ such that for all N, T

1. For each t,

$$E\|\frac{1}{\sqrt{NT}}\sum_{s=1}^{T}\sum_{k=1}^{N}F_t[e_{ks}e_{kt}-E(e_{ks}e_{kt})]\|^2 \leq M$$

2. The $k \times k$ matrix satisfies

$$E\|\frac{1}{\sqrt{NT}}\sum_{t=1}^{T}\sum_{k=1}^{N}F_t\lambda_k' e_{kt}\|^2 \leq M$$

3. For each t, as $N \rightarrow N$,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i e_{it} \xrightarrow{\text{d}} N(0,\Gamma_t)$$

where $\Gamma_t = \lim\limits_{N\rightarrow\infty}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda_j E(e_{it}e_{jt})$

4. For each i, as $T \rightarrow \infty$,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t e_{it} \xrightarrow{\text{d}} N(0,W_i),$$

where $W_i = \underset{T\rightarrow\infty}{plim}\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}E[F_tF_s' e_{is}e_{it}]$.

Assumption A is more general than that of classical factor analysis, as it allows F_t to be dynamic such that $A(L)F_t = v_t$. However, this dynamic does not enter into X_{it} directly, so the relationship between X_{it} and F_t remains static. Assumption B ensures that each factor has a non-trivial contribution to the variance of X_t. Assumption C allows for limited time series and cross section dependence in the idiosyncratic components as well as heteroscedasticity in both dimensions. Assumption D does not require independence to hold and is implied by Assumptions A and C. Throughout this article the number of factors (k) is assumed fixed as N and T grow. Assumptions A-D are sufficient for consistently estimating the number of factors (k), as well as the factors themselves and their corresponding loadings. Assumption E is a stronger version of assumptions C2 and C3 but still reasonable. For example, under time and cross sectional independence assumptions E1 and E2 become equivalent since $\frac{1}{N}\sum_{i=1}^{N}(e_{it})^2 \leq M$ and $E(e_{it})^2 \leq M$ which are both implied by assumption C1. Similarly, Assumption F is not stringent because sums in F1 and F2 involve zero mean random variables. The last two assumptions are central limit theorems, which are satisfied by several mixing processes.

6.2 Theorems & Proofs

Proof of Lemma 1. As outlined in the main text, in practice the true factors F in our information set are unobservable and have to be replaced by their estimated counterparts $\hat{F}$. Therefore,once $\hat{F}_{t-1}$ are replaced in I_{t-1} resulting in $\hat{I}_{t-1}$, we demonstrate how this factor estimation error affects the asymptotic properties of $S_{j,T}(\xi,\tau,\theta)$. The proof will be dealing with $S_{2,T}(\xi,\tau,\theta)$, where factor estimation error is present both within the model and the information set (thus both in the indicator function and the weighting exponential function) hence $m_j(\hat{I}_{t-1},\hat{\theta}(\tau)) = Y_{t-1}\hat{\theta}_{1,T}(\tau) - \hat{F}_{t-1}\hat{\theta}_{2,T}(\tau)$. It is easy to see that the results also hold for $S_{1,T}(\xi,\tau,\theta)$ where the factor estimation error is present only in the information set.

Our results will utilise the following identity, outlined in Bai & Ng (2002),

$$\hat{F}_t - H'F_t = V_{NT}^{-1}\big[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t) + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\zeta_{st} + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st} + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{st}\big] \tag{21}$$

where

$$\gamma_N(s,t) = E(\frac{1}{N}\sum_{s=1}^{N}e_{is}e_{it}) = E\big(\frac{e_s'e_t}{N}\big)$$

$$\zeta_{st} = \frac{e_s'e_t}{N} - \gamma_N(s,t)$$

$$\eta_{st} = \frac{F_s'\Lambda'e_t}{N}$$

$$\xi_{st} = \frac{F_t'\Lambda'e_s}{N}$$

We will also utilise the following lemmas proved in Stock & Watson (1999) and Bai & Ng (2002) respectively:

Lemma A.1: Under Assumptions A-D, as T, $N \to \infty$,

(i) $T^{-1}\hat{F}'(\frac{1}{TN}X'X)\hat{F} = V_{NT} \xrightarrow{p} V_{NT}$

(ii) $\frac{\hat{F}'F}{T}(\frac{\hat{\Lambda}'\Lambda}{N})\frac{F'\hat{F}}{T} \xrightarrow{p} V$

where V is the diagonal matrix consisting of the eigenvalues of $\Sigma_\Lambda\Sigma_F$.

Lemma A.2: Under assumptions A-D,

$$\delta_{NT}^2\left(\frac{1}{T}\sum_{t=1}^T \|\hat{F}_t - H'F_t\|^2\right) = O_p(1)$$

Also note that because V_{NT} converges to a positive definite matrix, by Lemma A.1, it follows that $\|V_{NT}\| = O_p(1)$. Furthermore, under assumptions A-B with $\hat{F}'F/T = I$ and Lemma A.1 , it is implied that $\|H\| = O_p(1)$.

Let hereinafter $\tilde{\epsilon}_t(\tau) = y_t - Y_{t-1}\hat{\theta}_{1,T}(\tau) - \hat{F}_{t-1}\hat{\theta}_{2,T}(\tau)$ be the residual when both factor estimation error and parameter estimation error is present. Also, let $\hat{\epsilon}_t(\tau) = y_t - Y_{t-1}\hat{\theta}_{1,T}(\tau) - F_{t-1}\hat{\theta}_{2,T}(\tau)$ be the residual with only parameter estimation error[5]. Therefore, the quantile empirical process could be expressed in the following way:

$$\begin{aligned}
S_{2,T}(\xi,\hat{\theta}(\tau)) &= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(y_t - Y_{t-1}\hat{\theta}_{1,T}(\tau) - \hat{F}_{t-1}\hat{\theta}_{2,T}(\tau) \le 0) - \tau]exp(\xi'\Phi(\hat{I}_{t-1}))\\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\tilde{\epsilon}_t(\tau) \le 0) - \tau]exp(\xi'\Phi(\hat{I}_{t-1}))\\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t(\tau) \le \hat{\epsilon}_t(\tau) - \tilde{\epsilon}_t(\tau)) - \tau]exp(\xi'\Phi(\hat{I}_{t-1}))
\end{aligned}$$

For simplicity, we are going to drop the dependence of ϵ and e_t on the specified quantile τ. By adding and subtracting $\mathbb{1}(\hat{\epsilon}_t \le 0)$ we obtain:

$$S_{2,T}(\xi,\hat{\theta}(\tau)) = \underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \le 0) - \tau]exp(\xi'\Phi(\hat{I}_{t-1}))}_{\text{Term I}} + \underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \le \hat{\epsilon}_t - \tilde{\epsilon}_t) - \mathbb{1}(\hat{\epsilon}_t \le 0)]exp(\xi'\Phi(\hat{I}_{t-1}))}_{\text{Term II}}$$

Focusing on **Term I**[6], using an intermediate value expansion around the true factors we obtain the

[5] Note that a similar residual is obtained under the QAR model and thus in the statistic $S_{1,T}(\xi,\hat{\theta}(\tau))$.

[6] It is worth mentioning here that **Term I** is equivalent to the term that would be obtained under $S_{1,T}(\xi,\hat{\theta}(\tau))$, where estimated factors only appear in the conditioning set and thus the indicator function only includes parameter estimation error. Solely proving that factor estimation error is negligible in **Term I** suffices to prove Lemma 1 for $S_{1,T}(\xi,\hat{\theta}(\tau))$.

following:

$$
\begin{aligned}
I &= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(y_t - Y_{t-1}'\theta(\tau) \leq 0) - \tau]exp(\xi'\Phi(\hat{I}_{t-1}))\\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\epsilon_t(\tau) \leq 0) - \tau]exp(\xi_Y'\Phi(Y_{t-1}))exp(\xi_F'\Phi(\hat{F}_{t-1}))\\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}\Psi_\tau(\epsilon_t)exp(\xi_Y'\Phi(Y_{t-1}))[exp(\xi_F'\Phi(HF_{t-1}) + \bigtriangledown_{F|\overline{HF}}(exp(\xi_F'\Phi(\hat{F}_{t-1})))[\hat{F}_{t-1} - HF_{t-1}]]\\
&= \underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}\Psi_\tau(\epsilon_t)exp(\xi_Y'\Phi(Y_{t-1}))exp(\xi_F'\Phi(HF_{t-1}))}_{I_1} \quad +\\
&\qquad \underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}\Psi_\tau(\epsilon_t)exp(\xi_Y'\Phi(Y_{t-1}))\bigtriangledown_{F|\overline{HF}}(exp(\xi_F'\Phi(\hat{F}_{t-1}))) * [\hat{F}_{t-1} - HF_{t-1}]}_{I_2}
\end{aligned}
$$

Term $\mathbf{I_1}$ is equivalent to a test statistic where the information set includes the true latent factors and thus only has parameter estimation error present[7] and is therefore the test statistic with no factor estimation error. Focusing then on the **Term $\mathbf{I_2}$**, using the identity of the forecast error as that is shown in Bai & Ng (2002) we get:

$$
\begin{aligned}
I_2 &= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \leq 0) - \tau]\ \bigtriangledown_{\hat{I}_{t-1}|\bar{I}_{t-1}}\ (exp(\xi'\Phi(\hat{I}_{t-1})))(\hat{F}_{t-1} - H'F_{t-1})\\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \leq 0) - \tau]\ \bigtriangledown_{\hat{F}_{t-1}|\overline{H'F}_{t-1}}\ (exp(\xi'\Phi(\hat{I}_{t-1})))\\
&\qquad\qquad * \Big\{V_{NT}^{-1}\Big[T^{-1}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t) + T^{-1}\sum_{s=1}^{T}\hat{F}_s\zeta_{st} + T^{-1}\sum_{s=1}^{T}\hat{F}_s\eta_{st} + T^{-1}\sum_{s=1}^{T}\hat{F}_s\xi_{st}\Big]\Big\}\\
&\leq \sup_t[\bigtriangledown_{\hat{F}_{t-1}|\overline{H'F}_{t-1}}(exp(\xi'\Phi(\hat{I}_{t-1})))]\underbrace{\|V_{NT}^{-1}\|}_{O_p(1)}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \leq 0) - \tau]\Big[\hat{F}_s\gamma_N(s,t) + \hat{F}_s\zeta_{st} + \hat{F}_s\eta_{st} + \hat{F}_s\xi_{st}\Big]
\end{aligned}
$$

Neglecting V_{NT}^{-1} and also $\sup_t[\bigtriangledown_{\hat{F}_{t-1}|\overline{H'F}_{t-1}}(exp(b'\Phi(\hat{I}_{t-1})))]$, which is bounded by definition and accounting

[7]This term is equivalent to the one in Escanciano & Velasco (2010), with the exception that the imaginary unit is used instead of the Borel measurable function, Φ.

for the fact that $T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2 = O_p(1)$, under assumption E1, the first term of **Term $\mathbf{I_2}$** is bounded by:

$$\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}|\mathbb{1}(\hat{\epsilon}_t \le 0)-\tau|\|\hat{F}_s\gamma_N(s,t)\| &\le \underbrace{sup_t|\mathbb{1}(\hat{\epsilon}_t \le 0)-\tau|}_{\le M \text{ by definition}} T^{-\frac{1}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\|\hat{F}_s\gamma_N(s,t)\| \\
&\le M * T^{-\frac{3}{2}}\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{s=1}^{T}\Big(\sum_{t=1}^{T}|\gamma_N(s,t)|\Big)^2\Big)^{\frac{1}{2}} \\
&= O_p\Big(\frac{1}{\sqrt{T}}\Big)
\end{aligned}$$

Similarly, under assumption C5 the second term is bounded by:

$$\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}|\mathbb{1}(\hat{\epsilon}_t \le 0)-\tau|\|\hat{F}_s\zeta_{st}\| &\le \underbrace{sup_t|\mathbb{1}(\hat{\epsilon}_t \le 0)-\tau|}_{\le M \text{ by definition}} T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\|\hat{F}_s\zeta_{st}\| \\
&\le M * \sqrt{T}\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{s=1}^{T}\Big(T^{-1}\sum_{t=1}^{T}|\zeta_{st}|\Big)^2\Big)^{\frac{1}{2}} \\
&\le M * \frac{\sqrt{T}}{\sqrt{N}}\underbrace{\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}}_{O_p(1)}\Big(T^{-1}\sum_{s=1}^{T}\underbrace{\Big(T^{-1}\sum_{t=1}^{T}|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{is}e_{it}-\gamma_N(s,t)|\Big)^2}_{O_p(1)}\Big)^{\frac{1}{2}} \\
&= O_p\Big(\frac{\sqrt{T}}{\sqrt{N}}\Big)
\end{aligned}$$

Furthermore, given that $T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2 = O_p(1)$, under assumptions A and F3, we can obtain an upper bound for the third and fourth term term:

$$\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}|\mathbb{1}(\hat{\epsilon}_t \le 0)-\tau|\|\hat{F}_s\eta_{st}\| &\le \underbrace{sup_t|\mathbb{1}(\hat{\epsilon}_t \le 0)-\tau|}_{\le M \text{ by definition}} T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\|\hat{F}_s\eta_{st}\| \\
&\le M * \sqrt{T}\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{s=1}^{T}\Big(T^{-1}\sum_{t=1}^{T}\|\frac{F_s'\Lambda' e_t}{N}\|\Big)^2\Big)^{\frac{1}{2}} \\
&\le M * \frac{\sqrt{T}}{\sqrt{N}}\underbrace{\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}}_{O_p(1)}\Big(\underbrace{(T^{-1}\sum_{s=1}^{T}\|F_s\|^2)}_{O_p(1)} * \underbrace{(T^{-1}\sum_{t=1}^{T}\|\frac{\Lambda' e_t}{\sqrt{N}}\|)^2)}_{O_p(1)}\Big)^{\frac{1}{2}} \\
&= O_p\Big(\frac{\sqrt{T}}{\sqrt{N}}\Big)
\end{aligned}$$

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}|\mathbb{1}(\hat{\epsilon}_t \leq 0)-\tau|\|\hat{F}_s\eta_{st}\| &\leq \underbrace{sup_t|\mathbb{1}(\hat{\epsilon}_t \leq 0)-\tau|}_{\leq M \text{ by definition}} T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\|\hat{F}_s\eta_{st}\| \\
&\leq M*\sqrt{T}\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{s=1}^{T}\Big(T^{-1}\sum_{t=1}^{T}\|\frac{F_t'\Lambda' e_s}{N}\|\Big)^2\Big)^{\frac{1}{2}} \\
&\leq M*\frac{\sqrt{T}}{\sqrt{N}}\Big(\underbrace{T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2}_{O_p(1)}\Big)^{\frac{1}{2}}\Big(\underbrace{(T^{-1}\sum_{s=1}^{T}\|\frac{\Lambda' e_s}{\sqrt{N}}\|^2)}_{O_p(1)}*\underbrace{(T^{-1}\sum_{t=1}^{T}\|F_t\|)^2}_{O_p(1)}\Big)\Big)^{\frac{1}{2}} \\
&= O_p\Big(\frac{\sqrt{T}}{\sqrt{N}}\Big)
\end{aligned}
$$

Overall therefore, given that $N \geqslant T$,

$$
\begin{aligned}
\textbf{Term I}_\mathbf{2} &= O_p\Big(\frac{1}{\sqrt{T}}\Big)+O_p\Big(\frac{\sqrt{T}}{\sqrt{N}}\Big)+O_p\Big(\frac{\sqrt{T}}{\sqrt{N}}\Big)+O_p\Big(\frac{\sqrt{T}}{\sqrt{N}}\Big) \\
&= O_p\Big(\frac{\sqrt{T}}{\min\{\sqrt{N},T\}}\Big)
\end{aligned}
$$

Moving on to **Term II:** By adding and subtracting $F(0)$ and $F(\hat{\epsilon}_t-\tilde{\epsilon})$, we obtain the following:

$$
\begin{aligned}
II &= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \leq \hat{\epsilon}_t-\tilde{\epsilon}_t)-\mathbb{1}(\hat{\epsilon}_t \leq 0)]exp(\xi'\Phi(\hat{I}_{t-1})) \\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}\Big\{[\mathbb{1}(\hat{\epsilon}_t \leq \hat{\epsilon}_t-\tilde{\epsilon}_t)-F(\hat{\epsilon}_t-\tilde{\epsilon})]-[T^{-\frac{1}{2}}\sum_{t=1}^{T}\mathbb{1}(\hat{\epsilon}_t \leq 0)-F(0)]\Big\}exp(\xi'\Phi(\hat{I}_{t-1})) \\
&\qquad + T^{-\frac{1}{2}}\sum_{t=1}^{T}[F(\hat{\epsilon}_t-\tilde{\epsilon})-F(0)]exp(\xi'\Phi(\hat{I}_{t-1})) \\
&\leq \sup_t \|exp(\xi'\Phi(\hat{I}_{t-1}))\|\Big\{T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t) \leq \hat{\epsilon}_t-\tilde{\epsilon}_t)-F(\hat{\epsilon}_t-\tilde{\epsilon})]-T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \leq 0)-F(0)]\Big\} \\
&\qquad + T^{-\frac{1}{2}}\sum_{t=1}^{T}[F(\hat{\epsilon}_t-\tilde{\epsilon})-F(0)]exp(\xi'\Phi(\hat{I}_{t-1})) \\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[F(\hat{\epsilon}_t-\tilde{\epsilon})-F(0)]exp(\xi'\Phi(\hat{I}_{t-1})+o_p(1)
\end{aligned}
$$

The last equality arises by stochastic equicontinuity and the bounded nature of the first component of the first term.

Similar to before, we can employ an intermediate value expansion on the remaining term and thus obtain the following:

$$
\begin{aligned}
&T^{-\frac{1}{2}}\sum_{t=1}^{T}[F(\hat{\epsilon}_t-\tilde{\epsilon})-F_{\hat{\epsilon}}(0)]exp(\xi'\Phi(\hat{I}_{t-1}))\\
&\quad=T^{-\frac{1}{2}}\sum_{t=1}^{T}\Big[F\Big((Y_t-Y_{t-1}'\hat{\theta}(\tau)-H'F_{t-1}'\hat{\beta}(\tau))-(Y_t-Y_{t-1}'\hat{\theta}(\tau)-\hat{F}_{t-1}'\hat{\beta}(\tau))\Big)-F(0)\Big]exp(\xi'\Phi(\hat{I}_{t-1}))\\
&\quad=T^{-\frac{1}{2}}\sum_{t=1}^{T}\Big[F\Big(\underbrace{(Y_t-Y_{t-1}'\hat{\theta}(\tau)-(H'F_{t-1})'\hat{\beta}(\tau))-(Y_t-Y_{t-1}'\hat{\theta}(\tau)-(H'F_{t-1})'\hat{\beta}(\tau))}_{=0}\Big)-F(0)\Big)\\
&\qquad+\nabla_{H'F}F\Big((Y_t-Y_{t-1}'\hat{\theta}(\tau)-(H'F_{t-1}')'\hat{\beta}(\tau))-(Y_t-Y_{t-1}'\hat{\theta}(\tau)-(\overline{H'F}_{t-1})'\hat{\beta}(\tau))\Big)(\hat{F}_{t-1}-H'F_{t-1})exp(\xi'\Phi(\hat{I}_{t-1}))\\
&\quad=T^{-\frac{1}{2}}\sum_{t=1}^{T}\nabla_{H'F}F\Big(-((H'F_{t-1})'-(\overline{H'F}_{t-1})')\hat{\beta}(\tau)\Big)(\hat{F}_{t-1}-H'F_{t-1})exp(\xi'\Phi(\hat{I}_{t-1}))\\
&\quad=\underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}\nabla_{H'F}F\Big(-((H'F_{t-1})'-(\overline{H'F}_{t-1})')\hat{\beta}(\tau)\Big)(\hat{F}_{t-1}-H'F_{t-1})exp(\xi'\Phi(I_{t-1}))}_{\text{Term II}_1}\\
&\qquad+\underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}\nabla_{H'F}F\Big(-((H'F_{t-1})'-(\overline{H'F}_{t-1})')\hat{\beta}(\tau)\Big)(\hat{F}_{t-1}-H'F_{t-1})\nabla_{H'F}\,exp(\xi'\Phi(\overline{I}_{t-1}))(\hat{F}_{t-1}-H'F_{t-1})}_{\text{Term II}_2}
\end{aligned}
$$

Focusing therefore on $\mathbf{Term\ II_1}$ employing the factor estimation error identity:

$$
\begin{aligned}
II_2&=T^{-\frac{1}{2}}\sum_{t=1}^{T}\nabla_{H'F}F\Big(-((H'F_{t-1})'-(\overline{H'F}_{t-1})')\hat{\beta}(\tau)\Big)*(\hat{F}_{t-1}-H'F_{t-1})\Big]exp(b'\Phi(I_{t-1}))\\
&\le\sup_t\Big(exp(\xi'\Phi(I_{t-1}))\Big)*\Big[T^{-\frac{1}{2}}\sum_{t=1}^{T}\underbrace{\nabla_{H'F}F\Big(-((H'F_{t-1})'-(\overline{H'F}_{t-1})')\hat{\beta}(\tau)\Big)}_{W_{t-1}}*(\hat{F}_{t-1}-H'F_{t-1})\Big]\\
&=\sup_t\Big(exp(\xi'\Phi(I_{t-1}))\Big)*T^{-\frac{1}{2}}\sum_{t=1}^{T}W_{t-1}*\Big\{V_{NT}^{-1}\big[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t)+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\zeta_{st}+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st}+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{st}\big]\Big\}\\
&=\sup_t\Big(exp(\xi'\Phi(I_{t-1}))\Big)*V_{NT}^{-1}\Big\{T^{-\frac{3}{2}}\sum_{t=1}^{T}W_{t-1}\hat{F}_s\gamma_N(s,t)+T^{-\frac{3}{2}}\sum_{t=1}^{T}W_{t-1}\hat{F}_s\zeta_{st}+T^{-\frac{3}{2}}\sum_{t=1}^{T}W_{t-1}\hat{F}_s\eta_{st}\\
&\qquad+T^{-\frac{3}{2}}\sum_{t=1}^{T}W_{t-1}\hat{F}_s\xi_{st}\Big\}
\end{aligned}
$$

where $E(\|W_{t-1}\|)\le M$ and $E(\|W_{t-1}\|^2)\le M$ by construction.

Taking the first term of $\mathbf{Term\ II_1}$, once again neglecting V_{NT}^{-1} and $\sup_t\Big(exp(\xi'\Phi(I_{t-1}))\Big)$ which are bounded by definition, then by adding and subtracting terms we obtain the following:

$$
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}\hat{F}_s\gamma_N(s,t)=T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}H'F_s\gamma_N(s,t)+T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}(\hat{F}_s-H'F_s)\gamma_N(s,t)
$$

The first component, under assumptions A and C2 is bounded by:

$$
\begin{aligned}
E[T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}H'F_s\gamma_N(s,t)] &= T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}E[W_{t-1}H'F_s\gamma_N(s,t)] \\
&\leq T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}(E\|W_{t-1}\|^2)^{\frac{1}{2}}(E\|F_s\|^2)^{\frac{1}{2}}|\gamma_N(s,t)| \\
&= M^2 * T^{-\frac{1}{2}} * T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}|\gamma_N(s,t)| \\
&= O_p(\frac{1}{\sqrt{T}})
\end{aligned}
$$

Meanwhile, under lemma A.2 and lemma 1i in Bai & Ng (2002):

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}(\hat{F}_s - H'F_s)\gamma_N(s,t) &\leq (T^{-1}\sum_{s=1}^{T}\|\hat{F}_s - H'F_s\|^2(T^{-1}\sum_{t=1}^{T}\|W_{t-1}\|^2)^{\frac{1}{2}}(T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}|\gamma_N(s,t)|^2)^{\frac{1}{2}} \\
&= O_p(\frac{1}{\delta_{NT}\sqrt{T}})
\end{aligned}
$$

Therefore the first term is an $O_p(\frac{1}{\sqrt{T}})$.

Focusing now on the second term, with a similar manipulation:

$$
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}\hat{F}_s\zeta_{st} = T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}H'F_s\zeta_{st} + T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}(\hat{F}_s - H'F_s)\zeta_{st}
$$

Under assumption F1, the first component is bounded by:

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}H'F_s\zeta_{st} &= \sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}F_s\Big(\frac{1}{N}\sum_{i=1}^{N}e_{is}e_{it} - \gamma_N(s,t)\Big) \\
&= H' * \frac{1}{\sqrt{N}}T^{-1}\sum_{t=1}^{T}W_{t-1}\frac{1}{\sqrt{NT}}\sum_{s=1}^{T}\sum_{i=1}^{N}F_s(e_{is}e_{it} - \gamma_N(s,t)) \\
&\leq H' * \frac{1}{\sqrt{N}} * \Big(T^{-1}\sum_{t=1}^{T}\|W_{t-1}\|^2\Big)^{\frac{1}{2}}\Big[T^{-1}\sum_{t=1}^{T}\Big(\frac{1}{\sqrt{NT}}\sum_{s=1}^{T}\sum_{i=1}^{N}F_s(e_{is}e_{it} - \gamma_N(s,t))\Big)^2\Big]^{\frac{1}{2}} \\
&= O_p(\frac{1}{\sqrt{N}})
\end{aligned}
$$

The second component, under lemma A.2 and assumption F1 is bounded by:

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}(\hat{F}_s-H'F_s)\zeta_{st} &= T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}(\hat{F}_s-H'F_s)\Big(\frac{1}{N}\sum_{i=1}^{N}e_{is}e_{it}-\gamma_N(s,t)\Big)\\
&\leq \frac{\sqrt{T}}{\sqrt{N}}(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s-H'F_s\|^2)^{\frac{1}{2}}\Big[T^{-1}\sum_{S=1}^{T}\Big(\frac{1}{\sqrt{NT}}\sum_{T=1}^{T}\sum_{i=1}^{N}W_{t-1}(e_{is}e_{it}-\gamma_N(s,t))\Big)^2\Big]^{\frac{1}{2}}\\
&\leq \frac{\sqrt{T}}{\sqrt{N}}(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s-H'F_s\|^2)^{\frac{1}{2}}\Big[T^{-1}\sum_{s=1}^{T}\Big(\frac{1}{T}\sum_{t=1}^{T}(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(e_{is}e_{it}-\gamma_N(s,t)))W_{t-1}\Big)^2\Big]^{\frac{1}{2}}\\
&= O_p(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}})
\end{aligned}
$$

Therefore, the second term is an $O_p(\frac{1}{\sqrt{N}})$.

Similarly,for the third term of **Term II$_1$**, under lemma A.2 and assumption F3, we obtain that:

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}H'F_s\eta_{st} &= \sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}H'F_s\Big(\frac{1}{N}\sum_{i=1}^{N}\lambda_iF_s'e_{it}\Big)\\
&= \Big(T^{-1}\sum_{s=1}^{T}F_sF_s'\Big)\frac{1}{\sqrt{T}N}\sum_{t=1}^{T}\sum_{i=1}^{N}W_{t-1}\lambda_ie_{it}\\
&\leq H'\Big(T^{-1}\sum_{s=1}^{T}F_sF_s'\Big)\frac{\sqrt{T}}{\sqrt{N}}\Big(T^{-1}\sum_{t=1}^{T}W_{t-1}(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_ie_{it}\Big)\\
&= O_p(\frac{\sqrt{T}}{\sqrt{N}})
\end{aligned}
$$

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}(\hat{F}_s-H'F_s)\eta_{st} &= T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}W_{t-1}(\hat{F}_s-H'F_s)\Big(\frac{F_s\Lambda e_t}{N}\Big)\\
&= \Big(T^{-1}\sum_{s=1}^{T}(\hat{F}_s-H'F_s)F_s\Big)\Big(\frac{1}{NT}\sum_{t=1}^{T}W_{t-1}\Lambda e_t\Big)*\sqrt{T}\\
&\leq \Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s-H'F_s\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{s=1}^{T}\|F_s\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}W_{t-1}(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_ie_{it}\Big)*\frac{\sqrt{T}}{\sqrt{N}}\\
&= O_p(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}})
\end{aligned}
$$

Thus the third term is an $O_p(\frac{\sqrt{T}}{\sqrt{N}})$.

Lastly, for the fourth term of **Term $\mathbf{II_1}$**, by lemma A.2 and assumptions A and F3, we obtain:

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T} W_{t-1}H'F_s\xi_{st} &= \sum_{t=1}^{T}\sum_{s=1}^{T} W_{t-1}H'F_s\Big(\frac{F_t'\Lambda' e_s}{N}\Big) \\
&= H'\Big(T^{-1}\sum_{s=1}^{T}\frac{F_s e_s'\Lambda}{N}\Big)\Big(T^{-1}\sum_{t=1}^{T} W_{t-1}F_t'\Big) * \sqrt{T} \\
&\leq \frac{\sqrt{T}}{\sqrt{N}} * \Big(T^{-1}\sum_{s=1}^{T}\|F_s\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}\|\frac{\Lambda e_t}{\sqrt{N}}\|^2\Big)^{\frac{1}{2}}\Big(\frac{1}{T}\sum_{t=1}^{T}\|W_{t-1}\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}\|F_t\|^2\Big)^{\frac{1}{2}} \\
&= O_p(\frac{\sqrt{T}}{\sqrt{N}})
\end{aligned}
$$

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T} W_{t-1}(\hat{F}_s - H'F_s)\xi_{st} &= T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T} W_{t-1}(\hat{F}_s - H'F_s)\Big(\frac{F_t'\Lambda' e_s}{N}\Big) \\
&= \Big[T^{-1}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\frac{\Lambda' e_s}{N}\Big]\Big[T^{-1}\sum_{t=1}^{T} W_{t-1}F_t\Big] * \sqrt{T} \\
&\leq \frac{\sqrt{T}}{\sqrt{N}} * \Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s - H'F_s\|^2\Big)^{\frac{1}{2}} * \Big(T^{-1}\sum_{s=1}^{T}\|\frac{\Lambda' e_s}{N}\|^2\Big)^{\frac{1}{2}}\Big(\frac{1}{T}\sum_{t=1}^{T}\|W_{t-1}\|^2\Big)^{\frac{1}{2}} \\
&\qquad \underbrace{\Big(T^{-1}\sum_{t=1}^{T}\|F_t\|^2\Big)^{\frac{1}{2}}}_{O_p(1)} \\
&= O_p(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}})
\end{aligned}
$$

Thus the last term is an $O_p(\frac{\sqrt{T}}{\sqrt{N}})$.

Overall therefore,

$$
\begin{aligned}
\textbf{Term II}_\mathbf{1} &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\triangledown_{H'F}F\Big(-((H'F)'_{t-1} - (\overline{H'F})'_{t-1})\hat{\beta}(\tau)\Big) * (\hat{F}_{t-1} - H'F_{t-1})\Big]exp(\xi'\Phi(I_{t-1})) \\
&= O_p(\frac{1}{\sqrt{T}}) + O_p(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}}) + O_p(\frac{\sqrt{T}}{\sqrt{N}}) + O_p(\frac{\sqrt{T}}{\sqrt{N}}) \\
&= O_p\Big(\frac{\sqrt{T}}{\min\{\sqrt{N}, T\}}\Big)
\end{aligned}
$$

Now focusing on **Term $\mathbf{II_2}$**. Because $(x+y+z+u)^2 \leq 4(x^2+y^2+z^2+u^2)$:

$$
\begin{aligned}
II_2 &= T^{-\frac{1}{2}} \sum_{t=1}^{T} \bigtriangledown_{H'F} F\Big(-((H'F)'_{t-1} - (\overline{H'F})'_{t-1})\hat{\beta}(\tau)\Big)(\hat{F}_{t-1} - H'F_{t-1}) \bigtriangledown_{H'F} exp(\xi'\Phi(\overline{I}_{t-1}))(\hat{F}_{t-1} - H'F_{t-1}) \\
&\leq \sup_t \Big[\bigtriangledown_{H'F} exp(\xi'\Phi(\overline{I}_{t-1}))\Big] T^{-\frac{1}{2}} \sum_{t=1}^{T} \bigtriangledown_{H'F} F\Big(-((H'F)'_{t-1} - (\overline{H'F})'_{t-1})\hat{\beta}(\tau)\Big)(\hat{F}_{t-1} - H'F_{t-1})'(\hat{F}_{t-1} - H'F_{t-1}) \\
&\leq \sup_t \Big[\bigtriangledown_{H'F} exp(\xi'\Phi(\overline{I}_{t-1}))\Big] T^{-\frac{1}{2}} \sum_{t=1}^{T} \|W_{t-1}\| \|\hat{F}_{t-1} - H'F_{t-1}\|^2 \\
&\leq M * T^{-\frac{1}{2}} \sum_{t=1}^{T} \Big\{ \|W_{t-1}\| * 4\Big[T^{-2} \| \sum_{s=1}^{T} \hat{F}_s \gamma_N s,t\|^2 + T^{-2} \| \sum_{s=1}^{T} \hat{F}_s \zeta_{st}\|^2 + T^{-2} \| \sum_{s=1}^{T} \hat{F}_s \eta_{st}\|^2 + T^{-2} \| \sum_{s=1}^{T} \hat{F}_s \xi_{st}\|^2 \Big]^8 \Big\} \\
&\leq 4M * T^{-\frac{5}{2}} \sum_{t=1}^{T} \Big\{ \|W_{t-1}\| * \Big[\| \sum_{s=1}^{T} \hat{F}_s \gamma_N s,t\|^2 + \| \sum_{s=1}^{T} \hat{F}_s \zeta_{st}\|^2 + \| \sum_{s=1}^{T} \hat{F}_s \eta_{st}\|^2 + \| \sum_{s=1}^{T} \hat{F}_s \xi_{st}\|^2 \Big] \Big\}
\end{aligned}
$$

Focusing on the first term of **Term $\mathbf{II_2}$**, following from the fact that $T^{-1}\sum_{s=1}^{T} \|\hat{F}_s\|^2 = O_p(1)$ and under assumptions E1, we obtain:

$$
\begin{aligned}
T^{-\frac{5}{2}} \sum_{t=1}^{T} \|W_{t-1}\| * \Big[\| \sum_{s=1}^{T} \hat{F}_s \gamma_N s,t\|^2 \Big] &\leq T^{-\frac{3}{2}} \Big(T^{-1} \sum_{t=1}^{T} \|W_{t-1}\|^2 \Big)^{\frac{1}{2}} \Big(T^{-1} \sum_{t=1}^{T} \Big[\| \sum_{s=1}^{T} \hat{F}_s \gamma_N s,t\|^2 \Big]^2 \Big)^{\frac{1}{2}} \\
&\leq T^{-\frac{1}{2}} \Big(T^{-1} \sum_{t=1}^{T} \|W_{t-1}\|^2 \Big)^{\frac{1}{2}} \Big(T^{-1} \sum_{t=1}^{T} \Big[\Big(T^{-1} \sum_{s=1}^{T} \|\hat{F}_s\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{s=1}^{T} |\gamma_N s,t|^2 \Big)^{\frac{1}{2}} \Big]^4 \Big)^{\frac{1}{2}} \\
&= O_p\Big(\frac{1}{\sqrt{T}} \Big)
\end{aligned}
$$

Meanwhile, the second term of **Term $\mathbf{II_2}$**, under assumption C5 is bounded by:

$$
\begin{aligned}
&T^{-\frac{5}{2}} \sum_{t=1}^{T} \|W_{t-1}\| * \Big[\| \sum_{s=1}^{T} \hat{F}_s \zeta_{st}\|^2 \Big] \\
&\qquad \leq T^{-\frac{3}{2}} \Big(T^{-1} \sum_{t=1}^{T} \|W_{t-1}\|^2 \Big)^{\frac{1}{2}} \Big(T^{-1} \sum_{t=1}^{T} \Big[\| \sum_{s=1}^{T} \hat{F}_s \zeta_{st}\|^2 \Big]^2 \Big)^{\frac{1}{2}} \\
&\qquad \leq \sqrt{T} \Big(T^{-1} \sum_{t=1}^{T} \|W_{t-1}\|^2 \Big)^{\frac{1}{2}} \Big(T^{-1} \sum_{t=1}^{T} \Big[\Big(T^{-1} \sum_{s=1}^{T} \|\hat{F}_s\|^2 \Big)^{\frac{1}{2}} \Big(T^{-1} \sum_{s=1}^{T} \|\zeta_{st}\|^2 \Big)^{\frac{1}{2}} \Big]^4 \Big)^{\frac{1}{2}} \\
&\qquad \leq \sqrt{T} \Big(T^{-1} \sum_{t=1}^{T} \|W_{t-1}\|^2 \Big)^{\frac{1}{2}} \Big(T^{-1} \sum_{t=1}^{T} \Big[\Big(T^{-1} \sum_{s=1}^{T} \|\hat{F}_s\|^2 \Big)^{\frac{1}{2}} \Big(T^{-1} \sum_{s=1}^{T} |N^{-1} \sum_{i=1}^{N} e_{is}e_{it} - E(e_{is}e_{it})|^2 \Big)^{\frac{1}{2}} \Big]^4 \Big)^{\frac{1}{2}} \\
&\qquad = O_p\Big(\frac{\sqrt{T}}{\sqrt{N}} \Big)
\end{aligned}
$$

The third term under assumption F3 is bounded by:

$$T^{-\frac{5}{2}}\sum_{t=1}^{T}\|W_{t-1}\|*\Big[\|\sum_{s=1}^{T}\hat{F}_s\gamma_N s,t\|^2\Big] \leq T^{-\frac{3}{2}}\Big(T^{-1}\sum_{t=1}^{T}\|W_{t-1}\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}\Big[\|\sum_{s=1}^{T}\hat{F}_s\eta_{st}\|^2\Big]^2\Big)^{\frac{1}{2}}$$

$$\leq T^{-\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}\|W_{t-1}\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}\Big[\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}\Big(\sum_{s=1}^{T}\|\eta_{st}\|^2\Big)^{\frac{1}{2}}\Big]^4\Big)^{\frac{1}{2}}$$

$$= O_p\Big(\frac{\sqrt{T}}{N}\Big)$$

Similarly, the fourth term of $\mathbf{Term\ II_2}$ is bounded by:

$$T^{-\frac{5}{2}}\sum_{t=1}^{T}\|W_{t-1}\|*\Big[\|\sum_{s=1}^{T}\hat{F}_s\xi_{st}\|^2\Big]$$

$$\leq T^{-\frac{3}{2}}\Big(T^{-1}\sum_{t=1}^{T}\|W_{t-1}\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}\Big[\|\sum_{s=1}^{T}\hat{F}_s\xi_{st}\|^2\Big]^2\Big)^{\frac{1}{2}}$$

$$\leq \sqrt{T}\Big(T^{-1}\sum_{t=1}^{T}\|W_{t-1}\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}\Big[\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{s=1}^{T}\|\xi_{st}\|^2\Big)^{\frac{1}{2}}\Big]^4\Big)^{\frac{1}{2}}$$

$$\leq \sqrt{T}\Big(T^{-1}\sum_{t=1}^{T}\|W_{t-1}\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}\Big[\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{s=1}^{T}\|\frac{F_t'\Lambda' e_s}{N}\|^2\Big)^{\frac{1}{2}}\Big]^4\Big)^{\frac{1}{2}}$$

$$\leq \sqrt{T}\Big(T^{-1}\sum_{t=1}^{T}\|W_{t-1}\|^2\Big)^{\frac{1}{2}}\Big(T^{-1}\sum_{t=1}^{T}\Big[\Big(T^{-1}\sum_{s=1}^{T}\|\hat{F}_s\|^2\Big)^{\frac{1}{2}}\Big(\|F_t\|^2 * N^{-1}T^{-1}\sum_{s=1}^{T}\|\frac{\Lambda' e_s}{\sqrt{N}}\|^2\Big)^{\frac{1}{2}}\Big]^4\Big)^{\frac{1}{2}}$$

$$= O_p\Big(\frac{\sqrt{T}}{N}\Big)$$

Overall therefore,

$$\mathbf{Term\ II_2} = T^{-\frac{1}{2}}\sum_{t=1}^{T}\bigtriangledown_{H'F}F\Big(-((H'F)_{t-1}' - (\overline{H'F})_{t-1}')\hat{\beta}(\tau)\Big)(\hat{F}_{t-1} - H'F_{t-1})\bigtriangledown_{H'F}exp(\xi'\Phi(\overline{I}_{t-1}))(\hat{F}_{t-1} - H'F_{t-1})$$

$$= O_p(\frac{1}{\sqrt{T}}) + O_p(\frac{\sqrt{T}}{\sqrt{N}}) + O_p(\frac{\sqrt{T}}{N}) + O_p(\frac{\sqrt{T}}{N})$$

$$= O_p\Big(\frac{\sqrt{T}}{\min\{\sqrt{N}, T\}}\Big).$$

This concludes the proof which demonstrates that factor estimation error does not influence the test statistic, since uniformly in τ:

$$\begin{aligned}
S_{2,T}(\xi,\hat{\theta}(\tau)) &= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\tilde{\epsilon}_t \leq 0)-\tau]exp(\xi'\Phi(\hat{I}_{t-1}))\\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \leq 0)-\tau]exp(\xi'\Phi(I_{t-1})) + \boldsymbol{I_2} + \boldsymbol{II_1} + \boldsymbol{II_2}\\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \leq 0)-\tau]exp(\xi'\Phi(I_{t-1})) + O_p\Big(\frac{\sqrt{T}}{\min\{\sqrt{N},T\}}\Big)\\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\hat{\epsilon}_t \leq 0)-\tau]exp(\xi'\Phi(I_{t-1})) + o_p(1)
\end{aligned}$$

Proof of Theorem 1. This theorem establishes the limit distribution of $S_{j,T}(\xi,\hat{\theta}(\tau))$. Under the null hypothesis $H_{0,j}$ and Assumptions A-F, factor estimation error is negligible by Lemma 1. The theorem then follows from the fact that, under the additional assumptions G-J,

$$\sup_{\xi\in\Upsilon,\tau\in\mathcal{T}}|S_{j,T}(\xi,\hat{\theta}_T(\tau)) - S_{j,T}(\xi,\theta(\tau)) + G'(\xi,\theta(\tau))Q(\tau)| = o_p(1).$$

The statement demonstrates that the quantile-marked empirical process converges to a test statistic where parameter estimation error is not present and a component accounting for the fact that in practice $\theta(\tau)$ is unknown and has to be estimated from a sample $\{(Y_t, I'_{t-1})' : 1 \leq t \leq T\}$ by an estimator $\hat{\theta}_T(\tau)$. See Theorem 2 in Escanciano & Velasco (2010).

Under the null hypothesis and assumption G1, given that $S_{j,T}(\xi,\theta(\tau))$ is a zero-mean-square-integrable martingale for each $\nu = (\xi',\tau)' \in \Pi$, with $\Pi = \Upsilon \times \mathcal{T}$, the finite dimensional distributions of $S_{j,T}(\xi,\theta(\tau))$ weakly converge in the space $l^\infty(\Pi)$ to $S_{j,\infty}(\xi,\theta(\tau))$, a multivariate normal distribution with a zero mean vector and a covariance function given by:

$$Cov_\infty(\nu_1,\nu_2) = (\min\{\tau_1,\tau_2\} - \tau_1\tau_2)E[exp((\xi_1-\xi_2)'\Phi(I_0))] \tag{22}$$

where, $\nu_1 = (\xi'_1,\tau_1)'$ and $\nu_2 = (\xi'_2,\tau_2)'$ are generic elements of $\Upsilon\times\mathcal{T}$ (See Theorem 1 in Escanciano & Velasco (2010))). It is immediate to see here that the covariance function does not contain any time dependent cross terms, as under the null $\mathbb{1}(y_t - m_j(I_{t-1},\hat{\theta}(\tau)) \leq 0) - \tau$ is a martingale difference sequence.

Given also that under Assumption J, the quantile limit process $Q(\tau)$ is a zero mean Gaussian Process with a covariance function given by equation (12), the result then follows from the interaction of two separate Gaussian Processes.

6.3 Dataset

NR	FAME CODE	Series Name
1	D7BT	Consumer Price Index: all items
2	ABMI	Gross Domestic Product: chained volume measures
3	ABJR	Household final consumption expenditure
4	CKYY	IOP: Industry D: Manufacturing
5	CKYZ	IOP: Industry E: Electricity, gas and water supply
6	CKZF	IOP: Industry DF: Manufacturing of food, drink and tobacco
7	CKZG	IOP: Industry DG: Manufacturing of chemicals and man-made fibres
8	GDBQ	ESA95 Output Index: F:Construction
9	GDQH	SA95 Output Industry: I: Transport storage and communication
10	GDQS	SA95 Output Industry: G-Q: Total
11	IKBK	Balance of Payments: Trade in Goods and Services: Total exports
12	IKBL	Balance of Payments: Imports: Total Trade in Goods and Services
13	NMRY	General Government: Final consumption expenditure
14	NPQT	Total Gross Fixed Capital Formation
Household final consumption expenditure: durable goods		
15	ATQX	Furniture and households
16	ATRD	Carpets and other floor coverings
17	ATRR	Telephone and telefax equipment
18	ATRV	Audio visual equipment
19	ATRZ	Photo and cinema equipment and optical instruments
20	ATSD	Information processing equipment
21	LLKX	All funrishing and household
22	LLKY	All health
23	LLKZ	All transport
24	LLLA	All communication
25	LLLB	All recreation and culture
26	LLLC	All miscellaneous
27	TMMI	All purchases of vehicles
28	TMML	Motor cars
29	TMMZ	Motor cycles
30	TMNB	Major durables for outdoor recreation
31	TMNO	Bicycles
32	UTID	Total
33	UWIC	Therapeutic appliances and equipment

34	XYJP	Major house appliances
35	XYJR	Major tools and equipment
36	XYJT	Musical instruments and major durables for indoor recreation
37	ZAYM	Jewelery, clocks and watches

Household final consumption expenditure: semi-durable goods

38	ATQV	Shoes and other footwear
39	ATRF	Household and textiles
40	ATRJ	Glassware, tableware and household utensils
41	ATSH	Recording media
42	ATSL	Games, toys and hobbies
43	ATSX	Other personal effects
44	AWUW	Motor vehicle spares
45	CDZQ	Books
46	LLLZ	All clothing and footwear
47	LLMC	All recreation and culture
48	LLMD	All miscellaneous
49	UTIT	Total
50	XYJN	Clothing materials
51	XYJO	Other articles of clothing and clothing accessories
52	XYJQ	Small eectric household appliances
53	XYJS	Small tools and miscellaneous accessories
54	XYJU	Equipment for sport, camping etc
55	XYJX	Electrical appliances for personal care
56	ZAVK	Garments

Household final consumption expenditure: non-durable goods

57	ATSP	Other products for personal care
58	ATUA	Materials for the maintenance and repair of the dwelling
59	AWUX	Gardens, plants and flowers
60	CCTK	Meat
61	CCTL	Fish
62	CCTM	Milk, cheese and eggs
63	CCTN	Oils and fats
64	CCTO	Fruit
65	CCTT	Coffee,tea and cocoa
66	CCTU	Mineral, water and soft drinks
67	CCTY	Vehicle fuels and lubricants
68	CCUA	Electricity
69	CDZY	Newspapers and periodicals
70	LLLL	All housing, water, electricity, gas and other fuels

71	LLLM	All furnishing and household goods
72	LLLN	All health
73	LLLO	All transport
74	LLLP	All recreation and culture
75	LLLQ	All miscellaneous
76	LTZA	Gas
77	LTZC	Liquid fuels
78	TTAB	Solid fuels
79	UTHW	Wine, cider and sherry
80	UTIL	Total
81	UTXP	Pharmaceutical Products
82	UTZN	Water supply
83	UUIS	Spirits
84	UUVG	Beer
85	UWBK	All food
86	UWBL	Bread and cereals
87	UWFD	Vegetables
88	UWFX	Sugar and sweet products
89	UWGH	Food products n.e.c.
90	UWGI	All non-alcoholic beverages
91	UWHO	Non-durable household goods
92	UWIB	Other medical products
93	UWKQ	Pets and related products
94	XYJV	Miscellaneous printed matter
95	XYJW	Stationary and drawing materials
96	ZAKY	All alcoholic beverages and otbacco
97	ZWUN	All food and non-alcoholic beverages
98	ZWUP	Tobacco
99	ZWUR	All elctricity, gas and other fuels

Household final consumption expenditure: services

100	AWUY	Repair and hire of footwear
101	AWUZ	Services for the maintenance and reair of the dwelling
102	AWVA	Vehicle maintenance and repair
103	AWVB	Railways
104	AWVC	Air
105	AWVD	Sea and inland waterway
106	AWVE	Other
107	CCUO	Imputed rentals of owner-occupiers
108	CCVA	Games of chance

109	CCVM	Postal services
110	CCVZ	Hairdressing salons and personal grooming establishments
111	GBFG	Actual rentals paid by tenants
112	GBFK	All imputed rentals for housing
113	GBFN	Other imputed rentals
114	LLLR	All clothing and footwear
115	LLLS	All housing, water, electricity, gas and other fuels
116	LLLT	All funrishing and household
117	LLLU	All health
118	LLLV	Total transport
119	LLLW	All communication
120	LLLX	All recreation and culture
121	LLLY	All miscellaneous
122	UTIP	Total
123	UTMH	Paramedical services
124	UTYF	Hospital services
125	UTYH	Life insurance
126	UTZX	Sewerage collection
127	UWHI	Clothing, repair and hire of clothing
128	UWHK	Refuse collection
129	UWHM	Repair of furniture, furnishings and floor coverings
130	UWHN	Repair of household appliances
131	UWIA	Domestic and household services
132	UWKO	Repair of audio-visual, ohoto and information processing equipment
133	UWKP	Maintenance of other major durables for recreation and culture
134	UWLD	Veterinary and other services
135	ZAVQ	All actual rentals for housing
136	ZAWG	All out-patient services
137	ZAWI	Medical services
138	ZAWK	Dental services
139	ZAWQ	Other vehicle services
140	ZAWS	All transport services
141	ZAWU	Road
142	ZAWY	Telephone and telefax services
143	ZAXI	All recreational and cultural services
144	ZAXK	Recreational and sporting activities
145	ZAXM	Cultural services
146	ZAXS	All restaurants and hotels
147	ZAXU	All catering services

148	ZAXW	Restaurants, cafes etc
149	ZAYC	Canteens
150	ZAYE	Accommodation services
151	ZAYO	Social protection
152	ZAYQ	All insurance
153	ZAYS	Insurance connected with the dwelling
154	ZAYU	Insurance connected with health
155	ZAYW	Insurance connected with transport
156	ZAZA	All financial services n.e.c.
157	ZAZC	All financial services other than FISIM
158	ZAZE	Other services n.e.c.
159	ZWUT	Education
		Deflators
160	ABJS	Consumption
161	FRAH	RPI: Total Food
162	ROYJ	Wages
163	YBGB	GDP Deflator
		Money Series
164	M4ISA	M4 Deposits PNFCs
165	M4OSA	M4 Deposits OFCs
166	M4PSA	M4 Deposits Households
167	MALISA	M4 Lending Total
168	MALOSA	M4 Lending PNFCs
169	MALPSA	M4 Lending Households
		Asset Prices
170		Real nationwide house prices
171	GDF Data	FTSE All Share Index
172	IMF Data	Nominal Effective Exchange Rate (NEER)
173	GDF Data	Pounds to Euro
174	GDF Data	Pounds to US dollar
175	GDF Data	Pounds to Canadian dollar
176	GDF Data	Pounds to Australian dollar

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